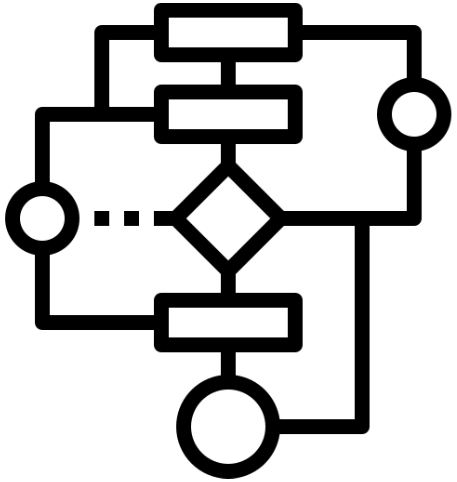


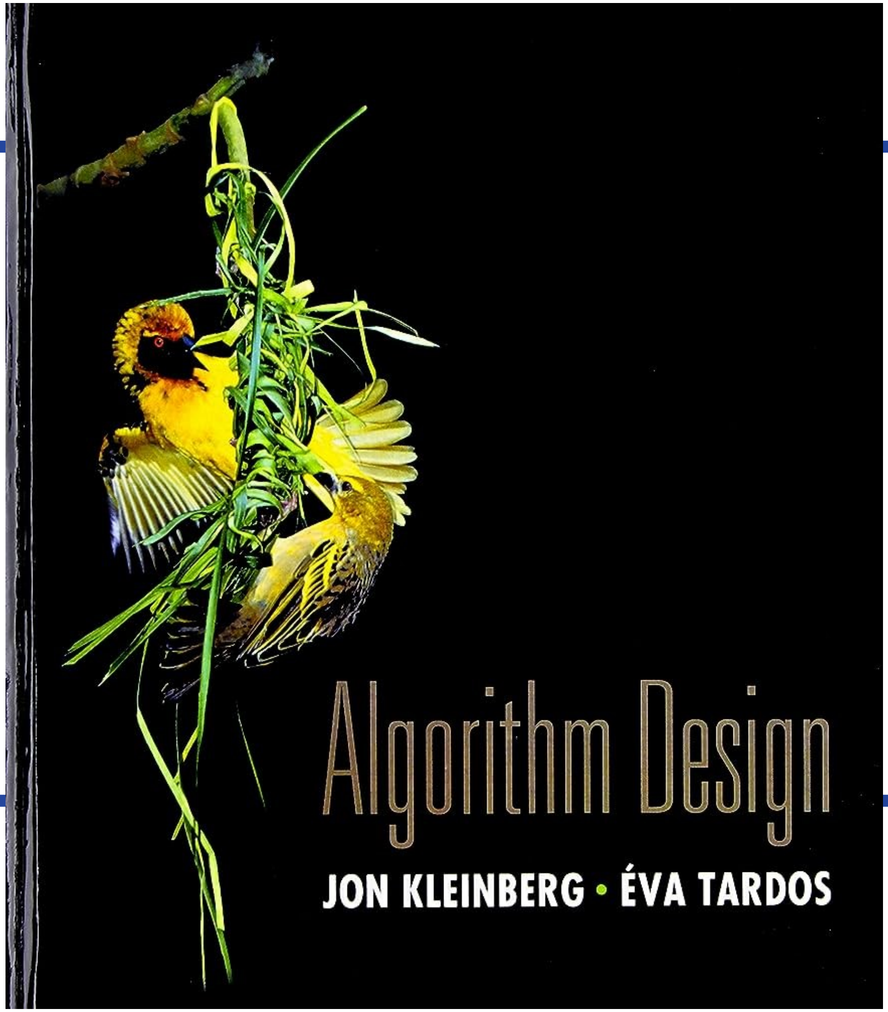
CS 310: Algorithms

Lecture 14

Instructor: Naveed Anwar Bhatti

Administrivia

- Three HWs (***Time Complexities***, ***Graphs*** and ***Divide & Conquer***)
- Solutions will be released on Friday
- Tutorial in A1 – Thursday, 6:00-7:30 pm
- ***Midterm Exam*** finalized



Chapter 4:
Greedy Algorithms

Greed is Goooood

Scenario: The Last of Us

Scenario: The Last of Us

Scene: We are in LUMS maze at night (limited visibility)

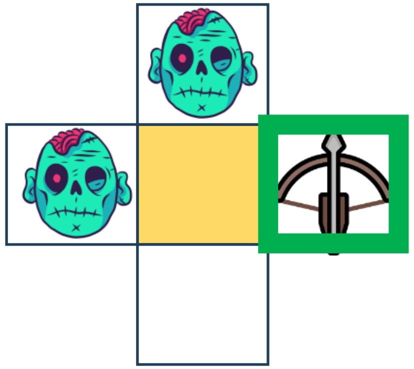
Objective: Collect the immediate weapon in sight and reach to safe zone

Rules:

- Can move in any direction: Up, Down, Left and Right
- Once a path is taken, there is no going back
- Game ends when you reach safe zone (or die)

Scenario 1

Scenario 1



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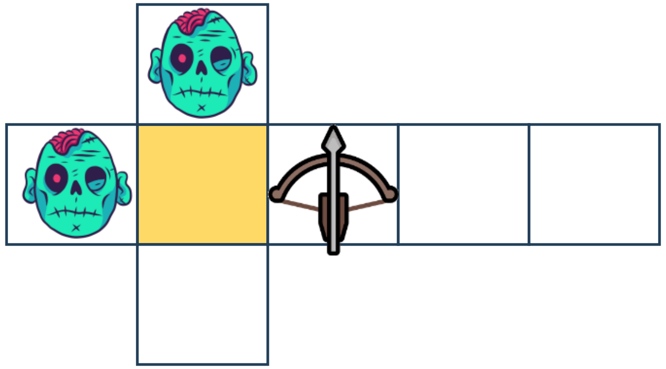

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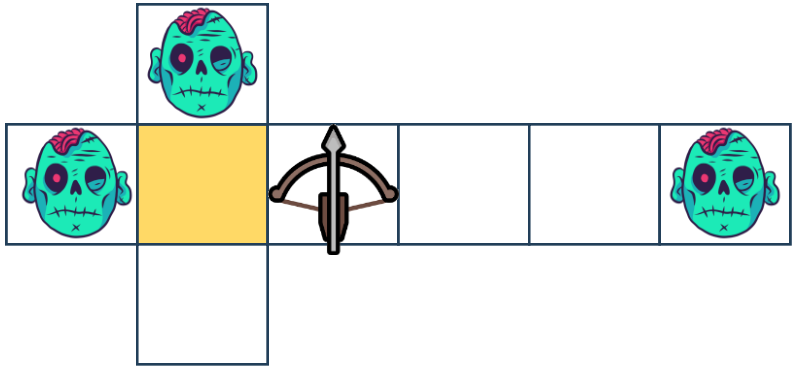

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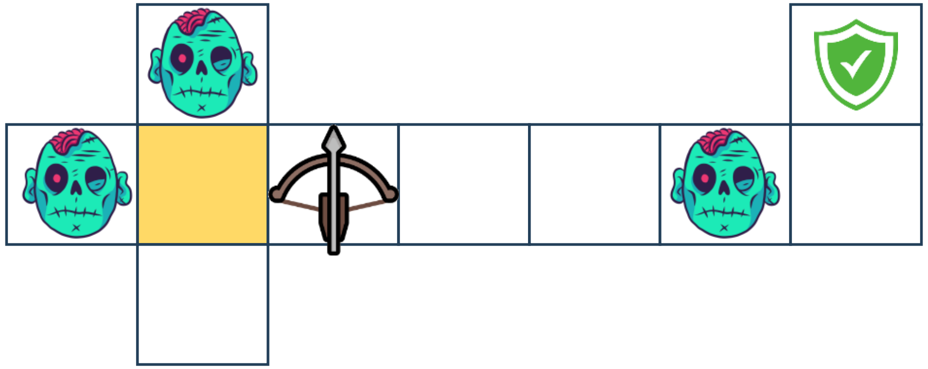

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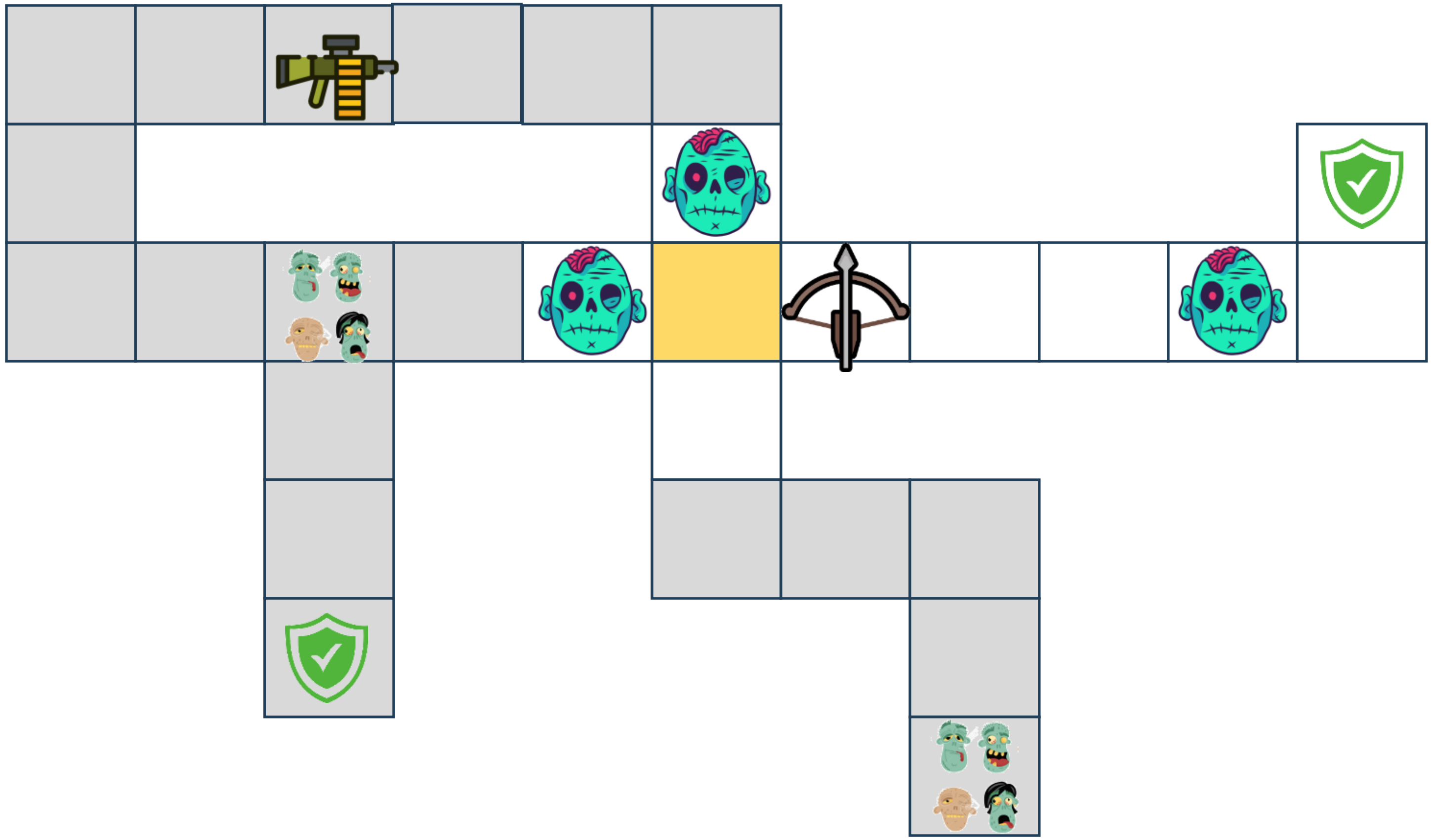

Scenario 2

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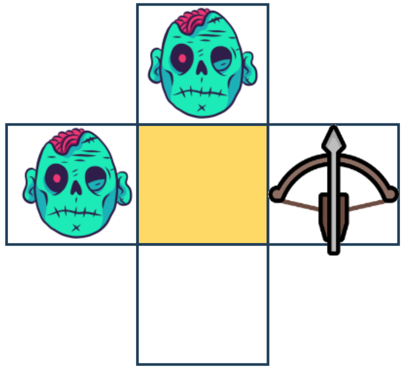

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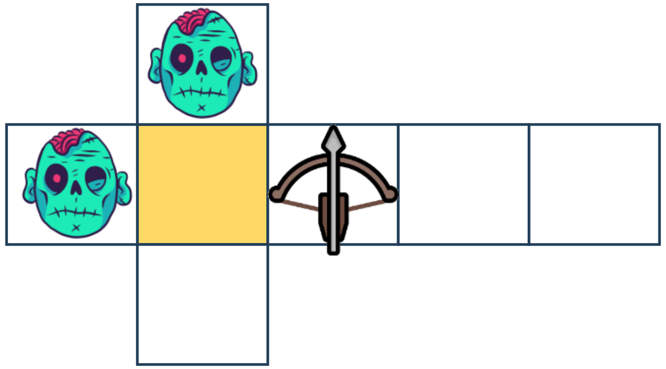

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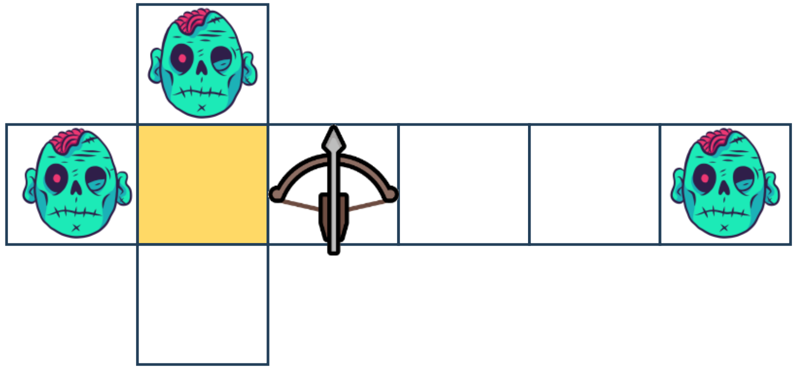

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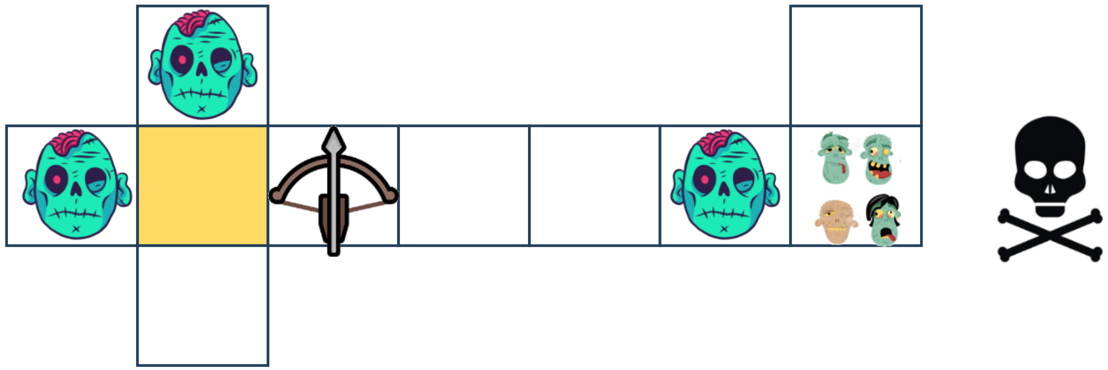

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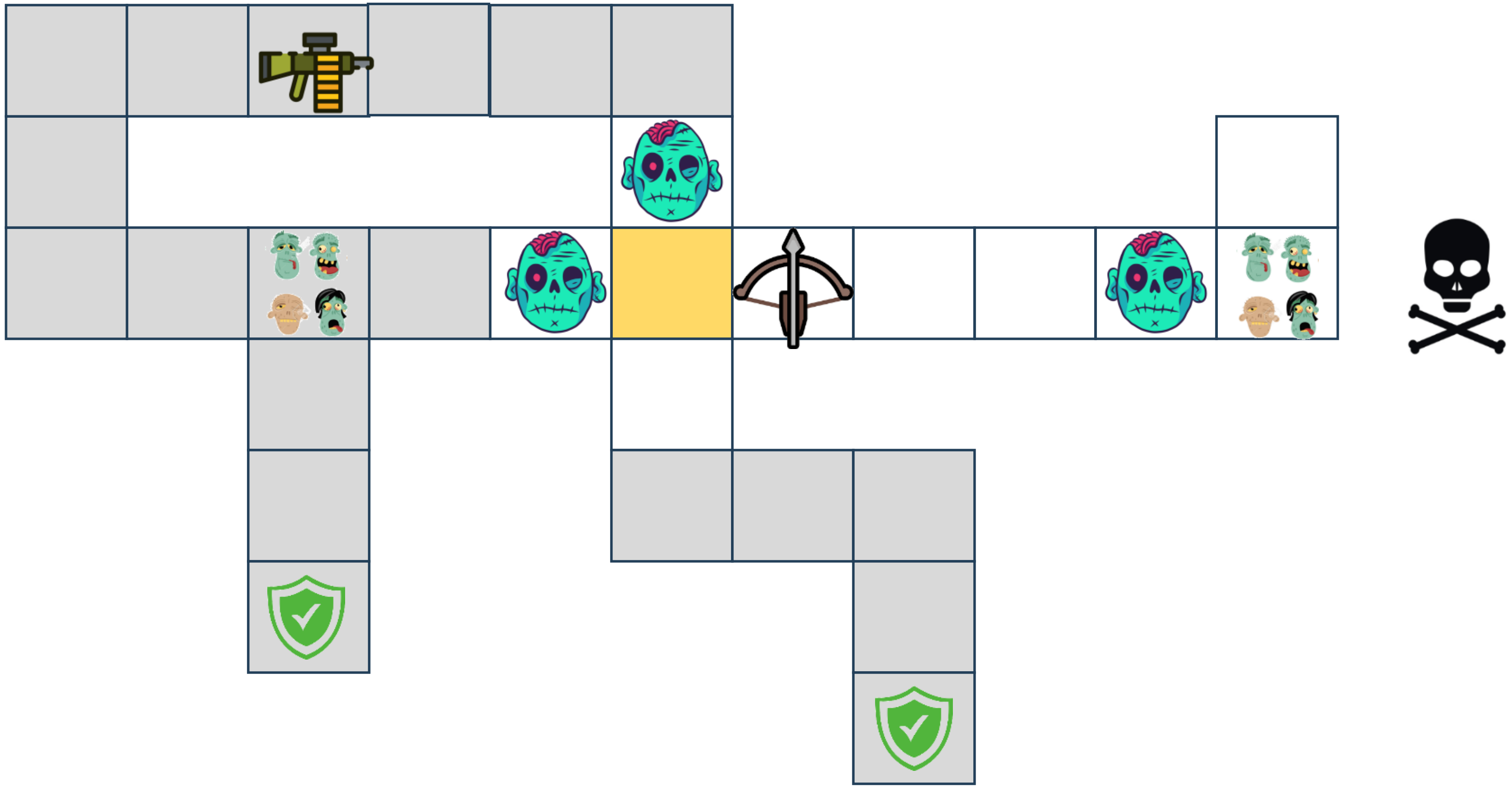


Greedy Algorithms

- An algorithm is greedy if it builds up solution in small steps

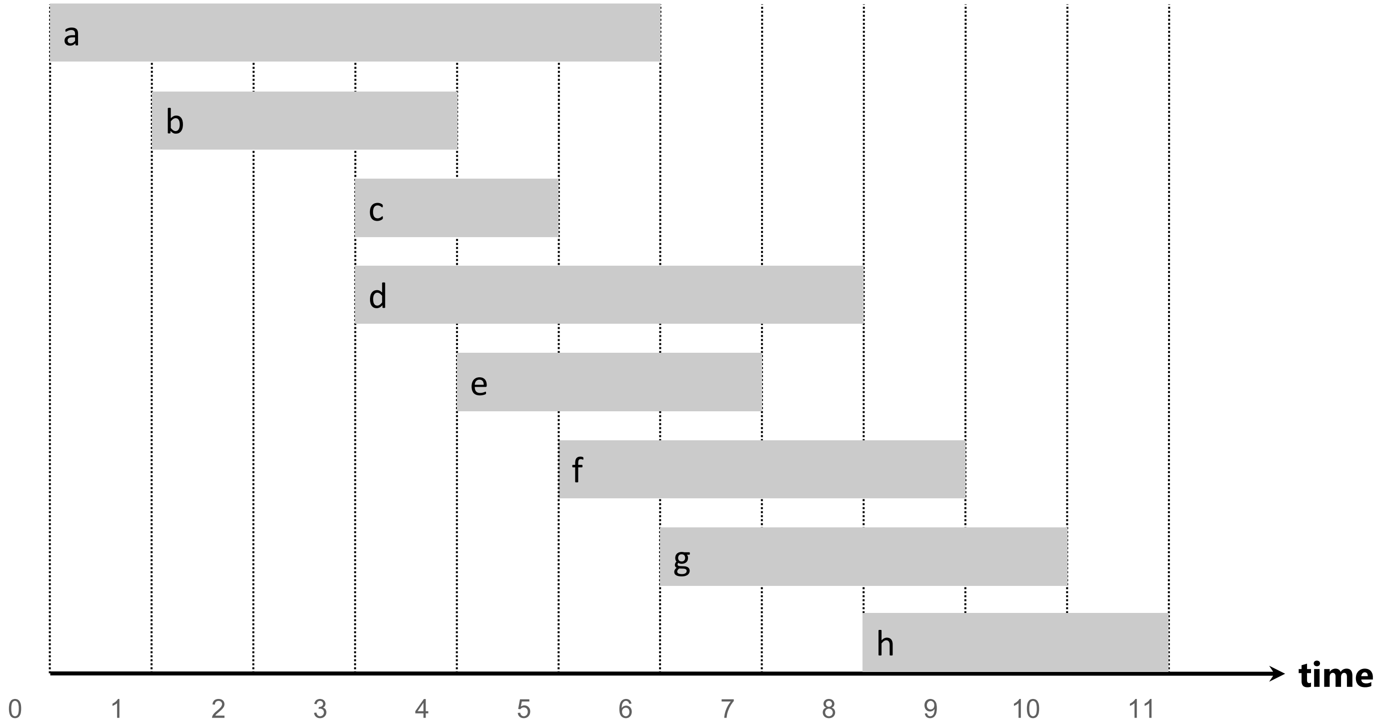
- In each step:
  - make a choice that looks best at the moment (greedy choice)
  - incrementally optimize the solution

- For many problems, greedy algorithms yield **global optimal** solution
  - Interval scheduling
  - Interval partitioning
  - Shortest paths in a graph
  - Minimum Spanning Tree

  - Perfect and stable matching

Interval Scheduling Problem

- Job j starts at s_j and finishes at f_j.
- Two jobs are compatible if they don't overlap.
- **Goal:** find maximum subset of mutually compatible jobs.



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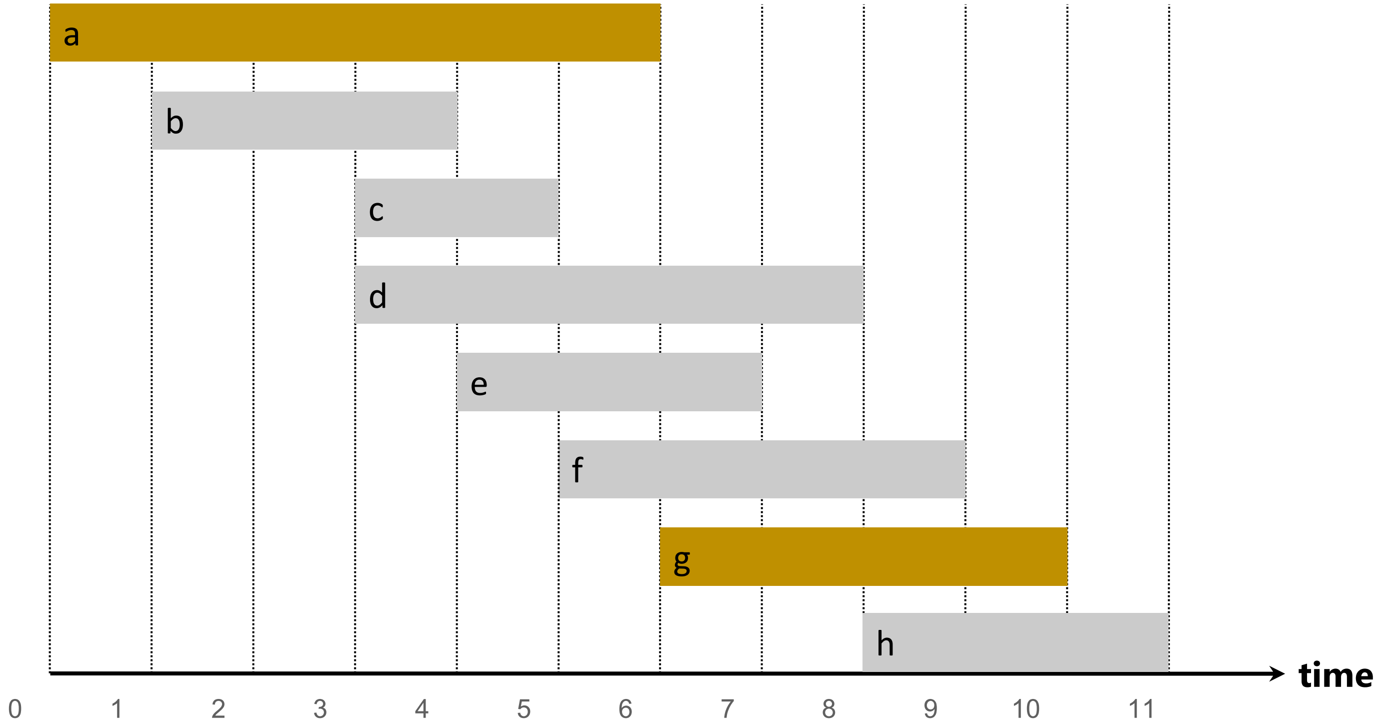

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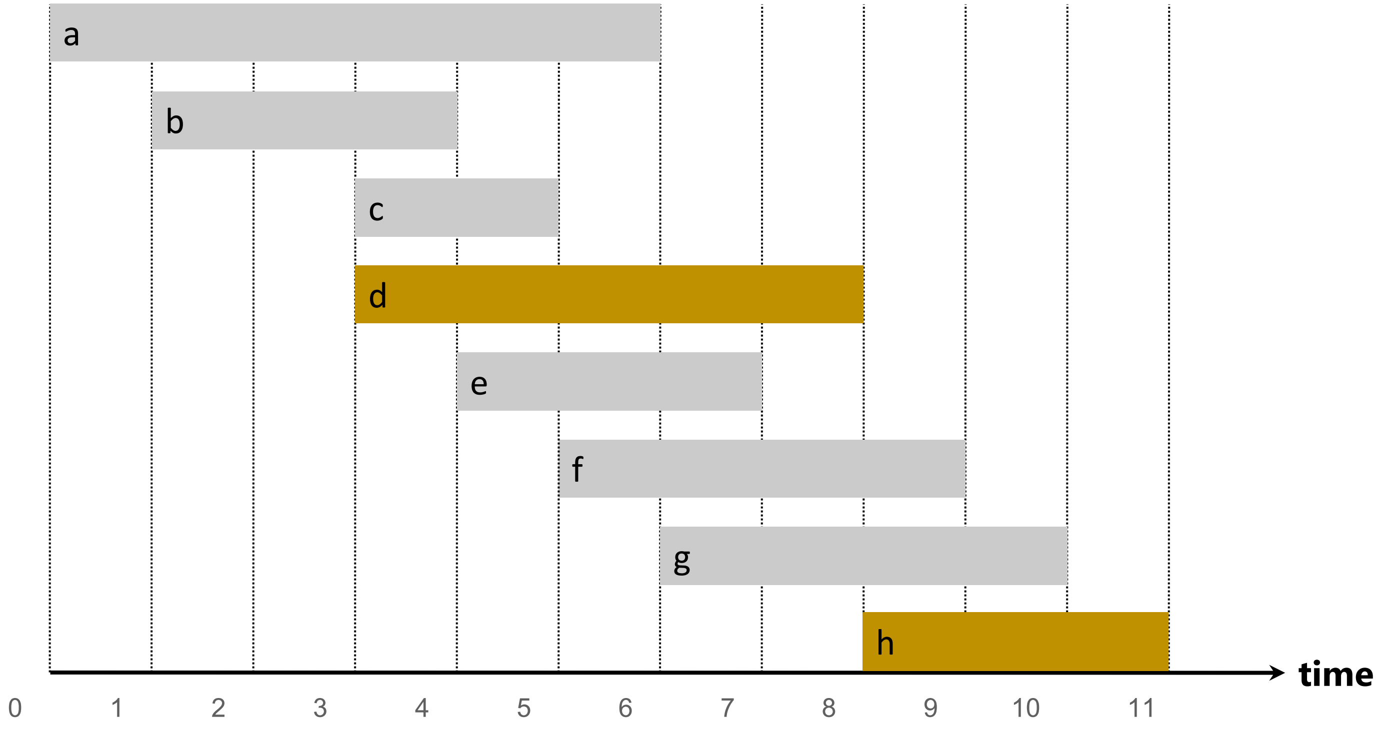

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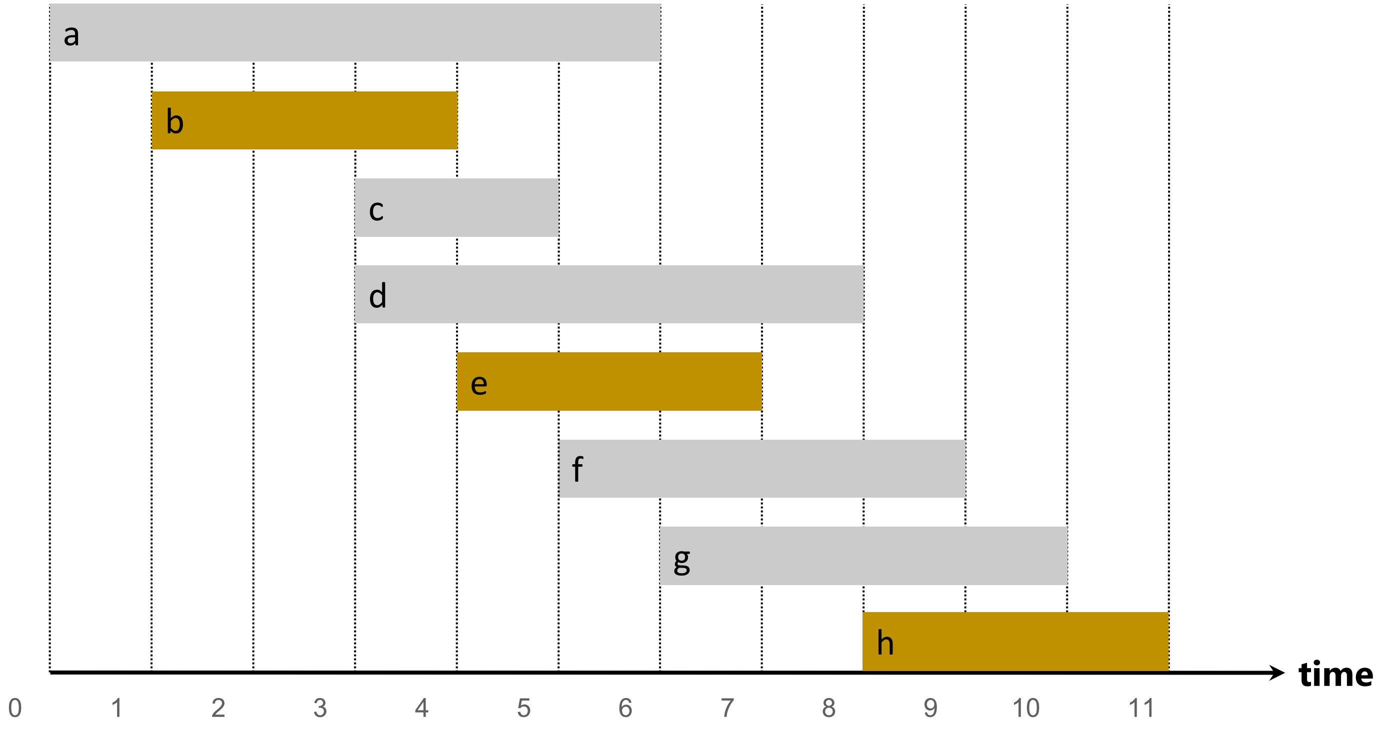
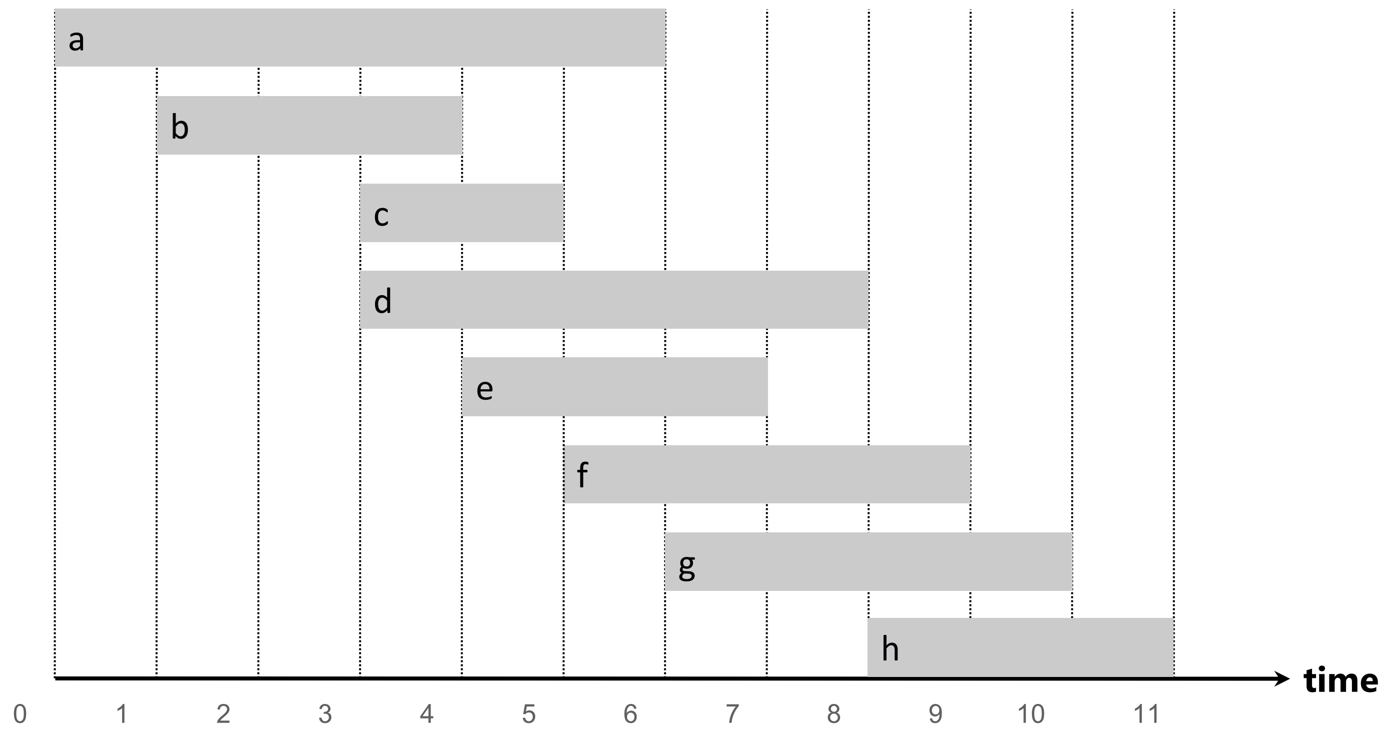


Interval Scheduling Problem

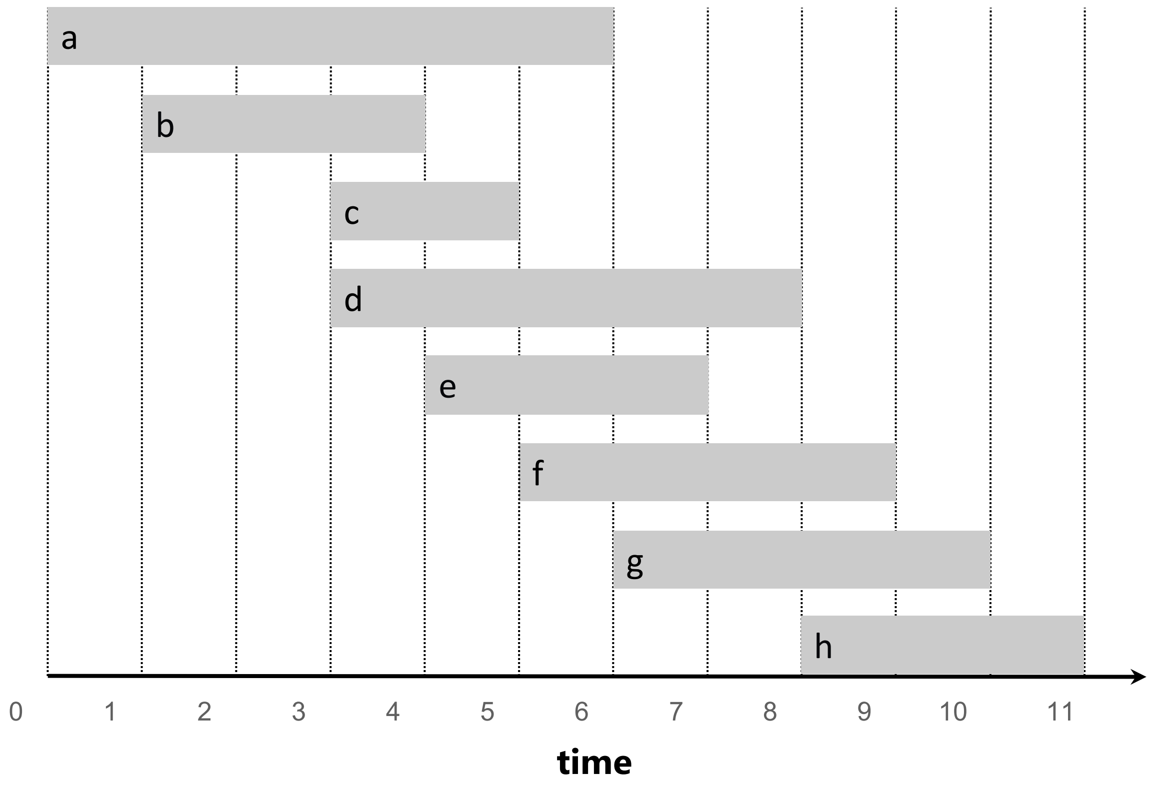
- Earliest Starting Request First
- Latest Finishing Request First
- Shortest Duration Request First
- Earliest Finish Time First

Sub-optimal



Interval Scheduling Problem

- **Earliest Starting Request First**
  - Select the request with the earliest start time
  - Eliminate any conflicting intervals (those that overlap with the chosen interval)
  - Continue this process until there are no more requests left.



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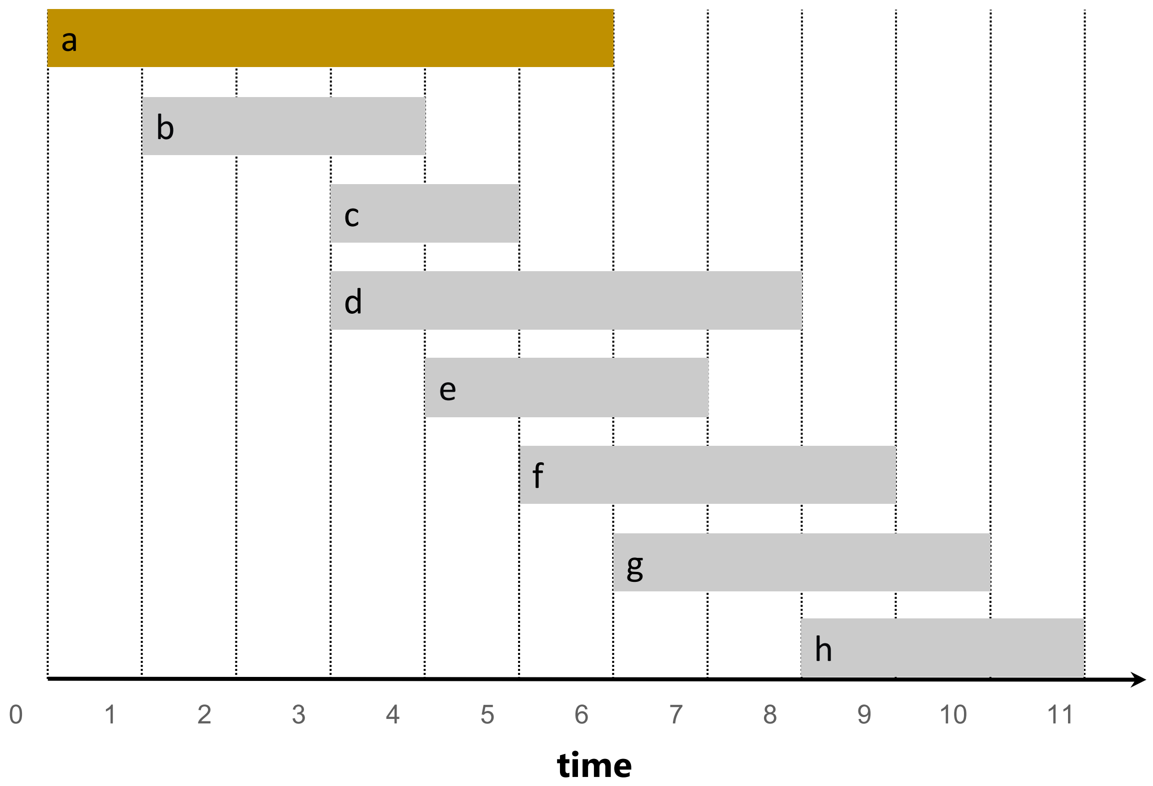

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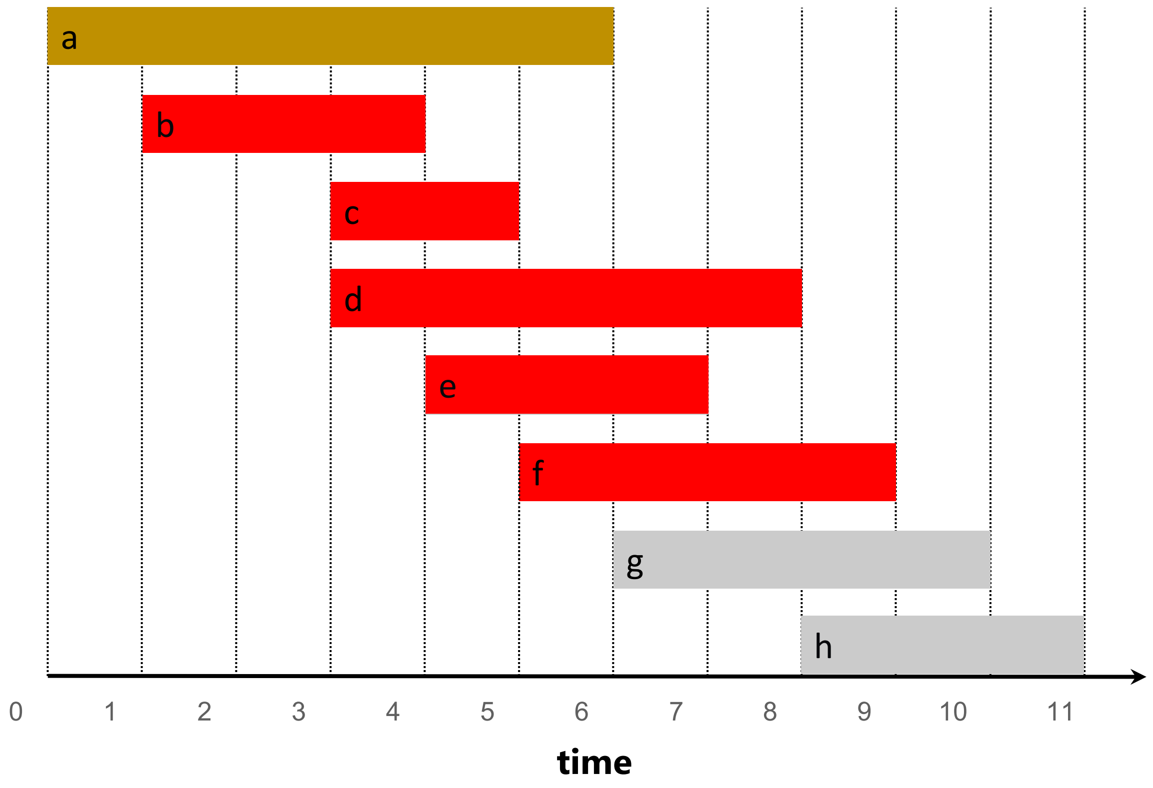

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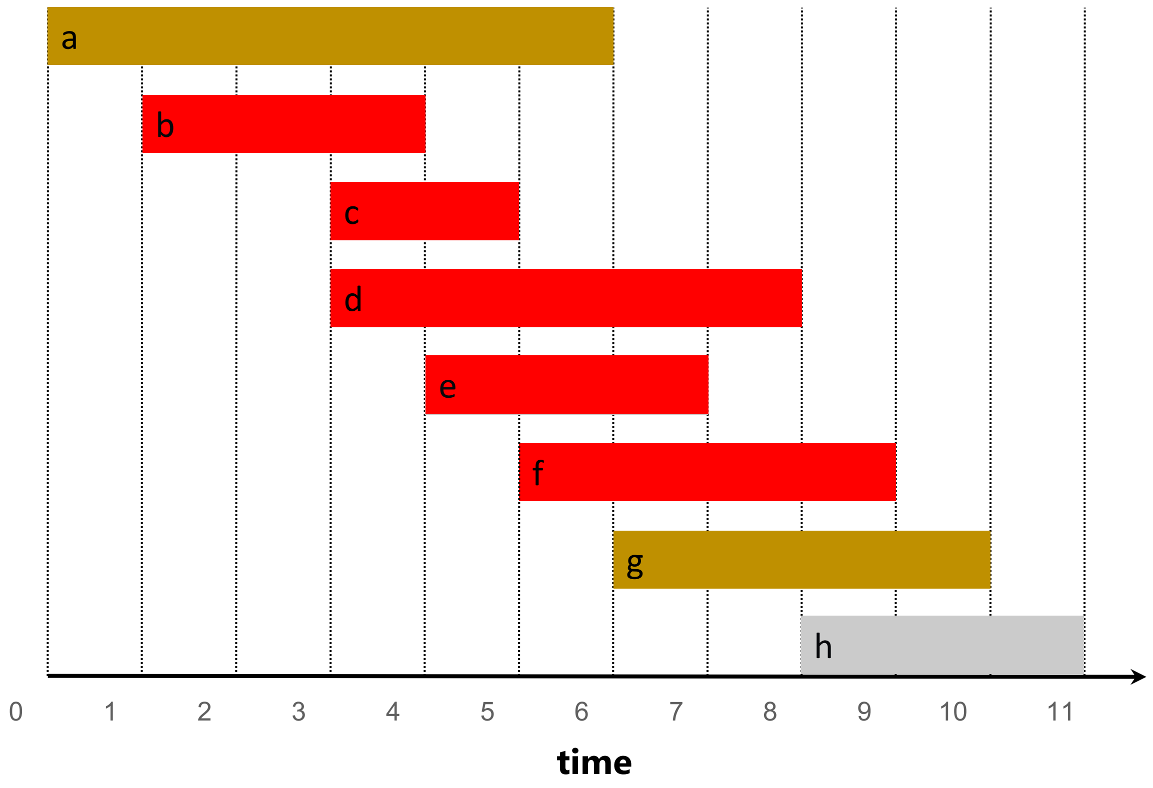

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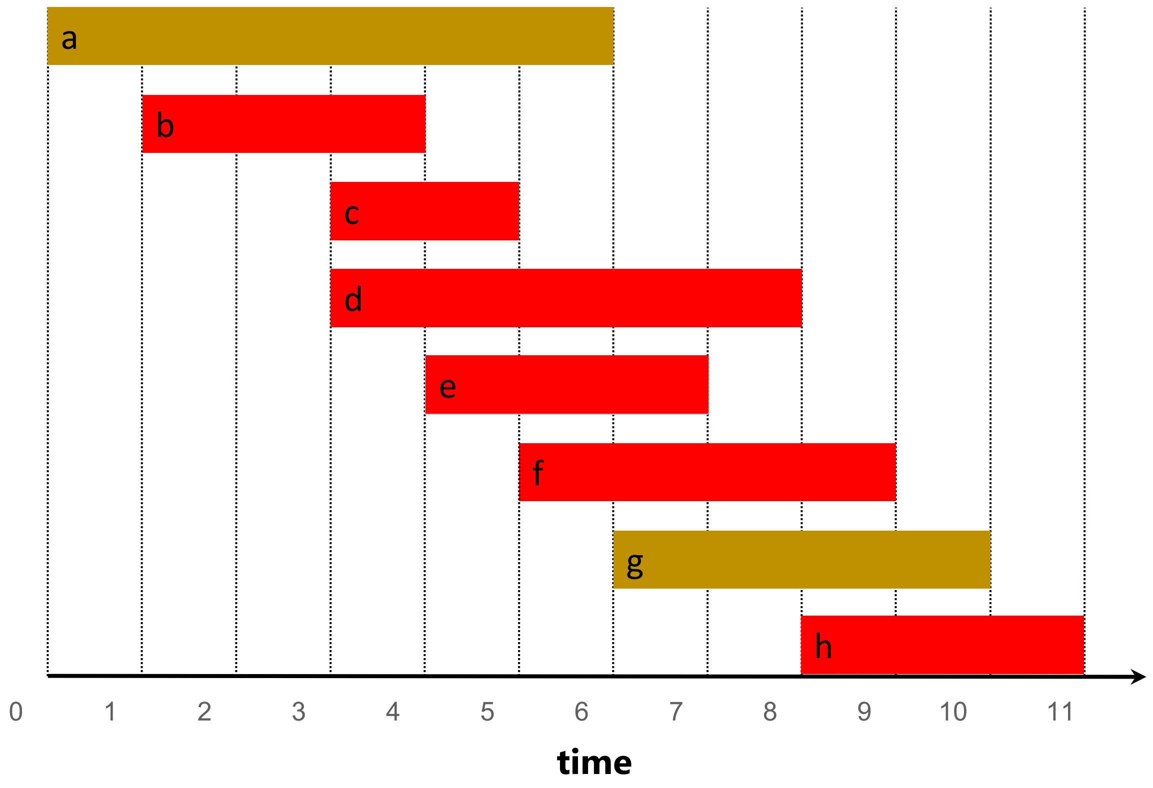

Interval Scheduling Problem

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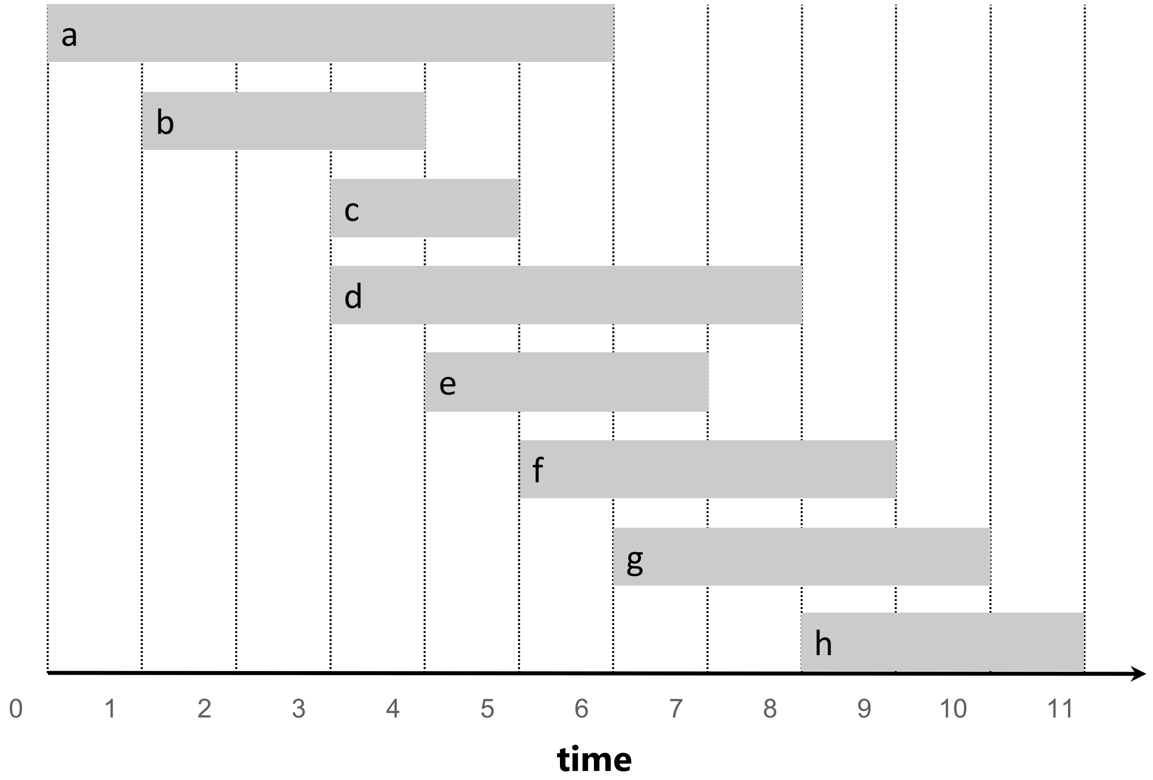

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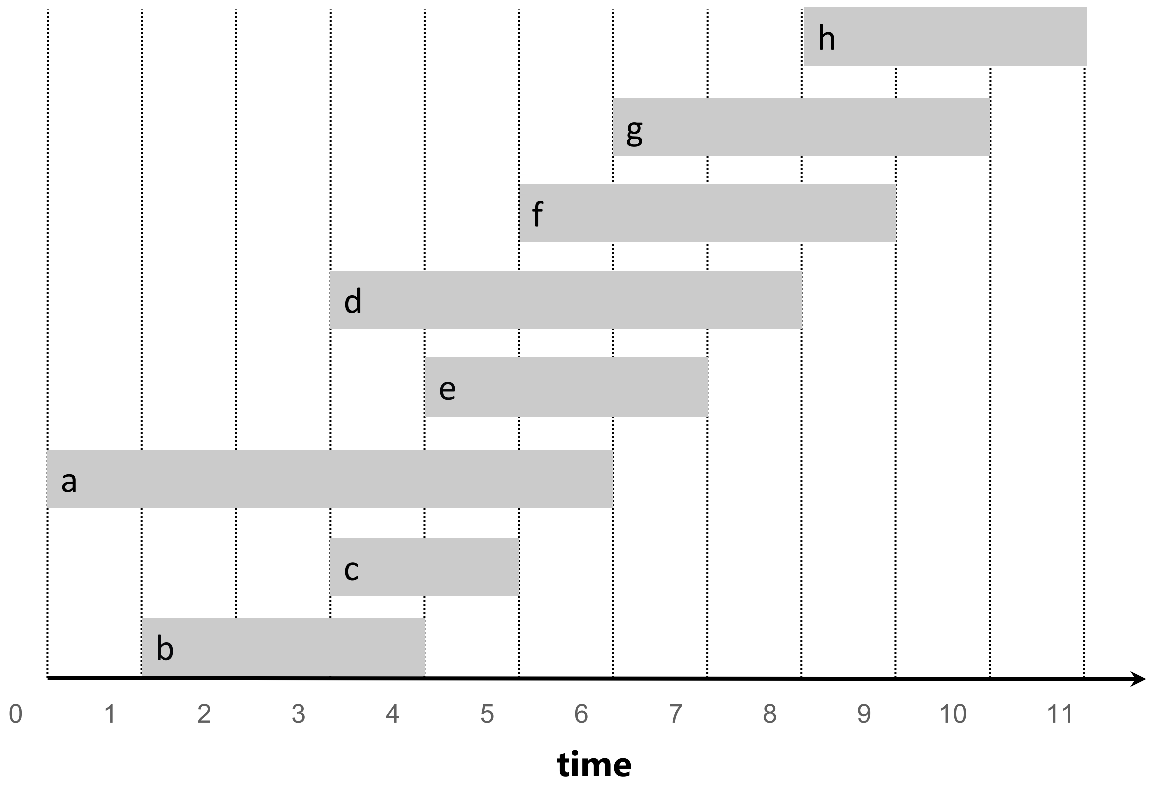

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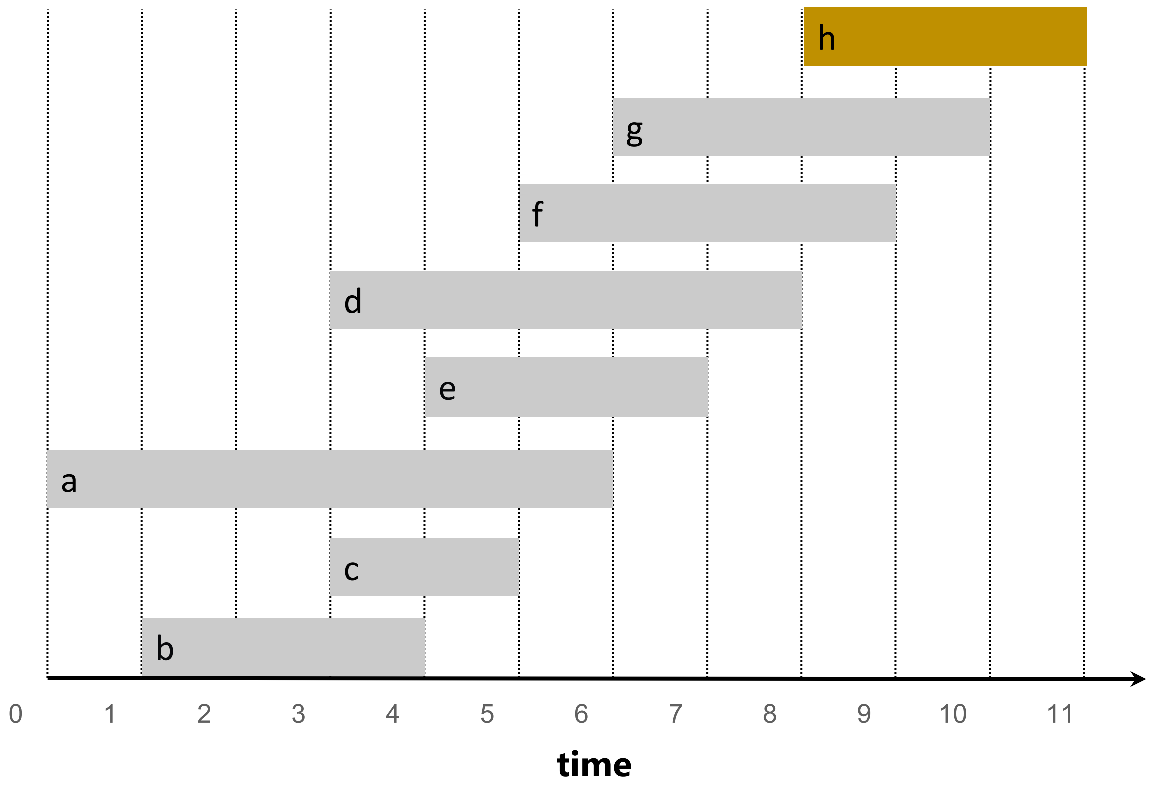

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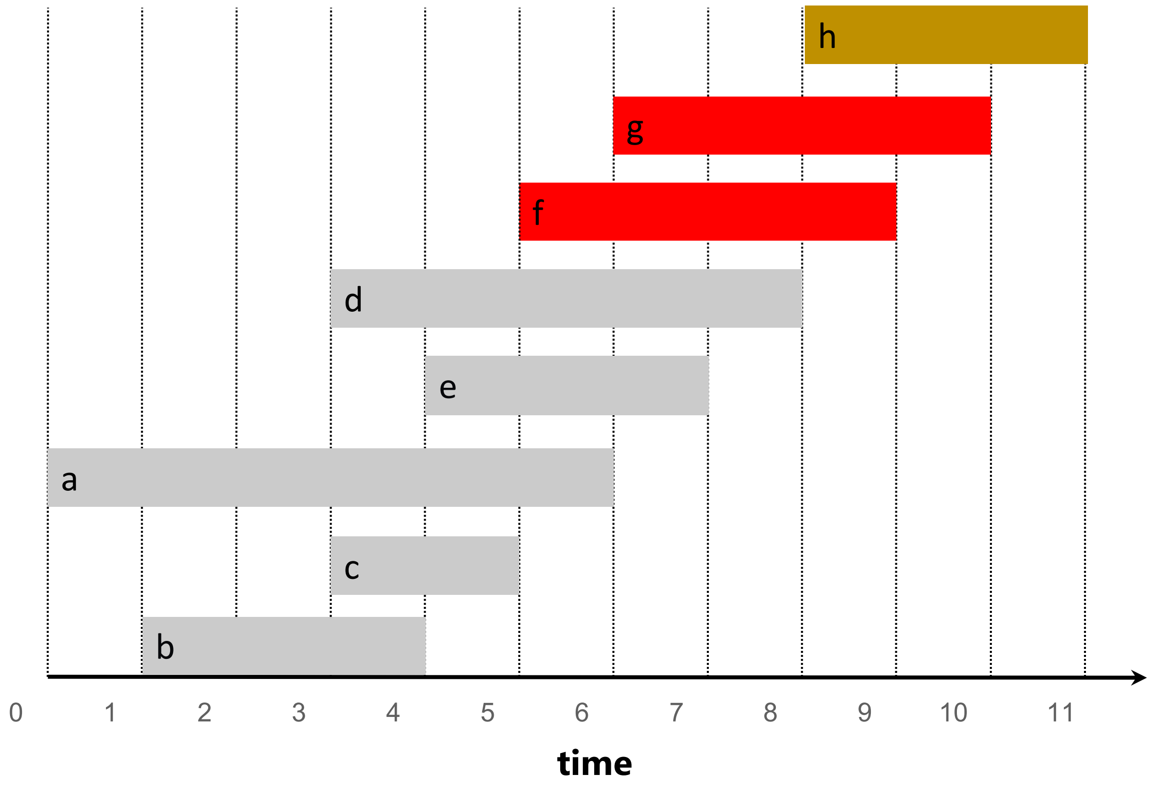

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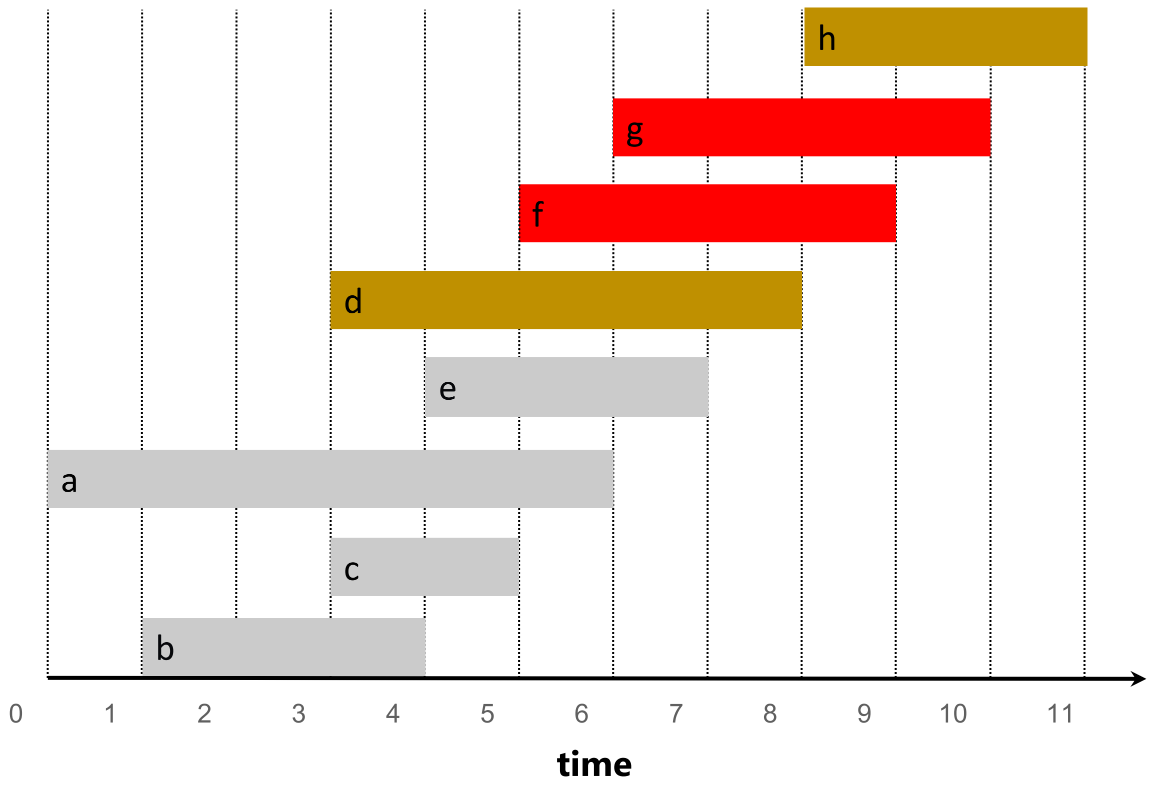

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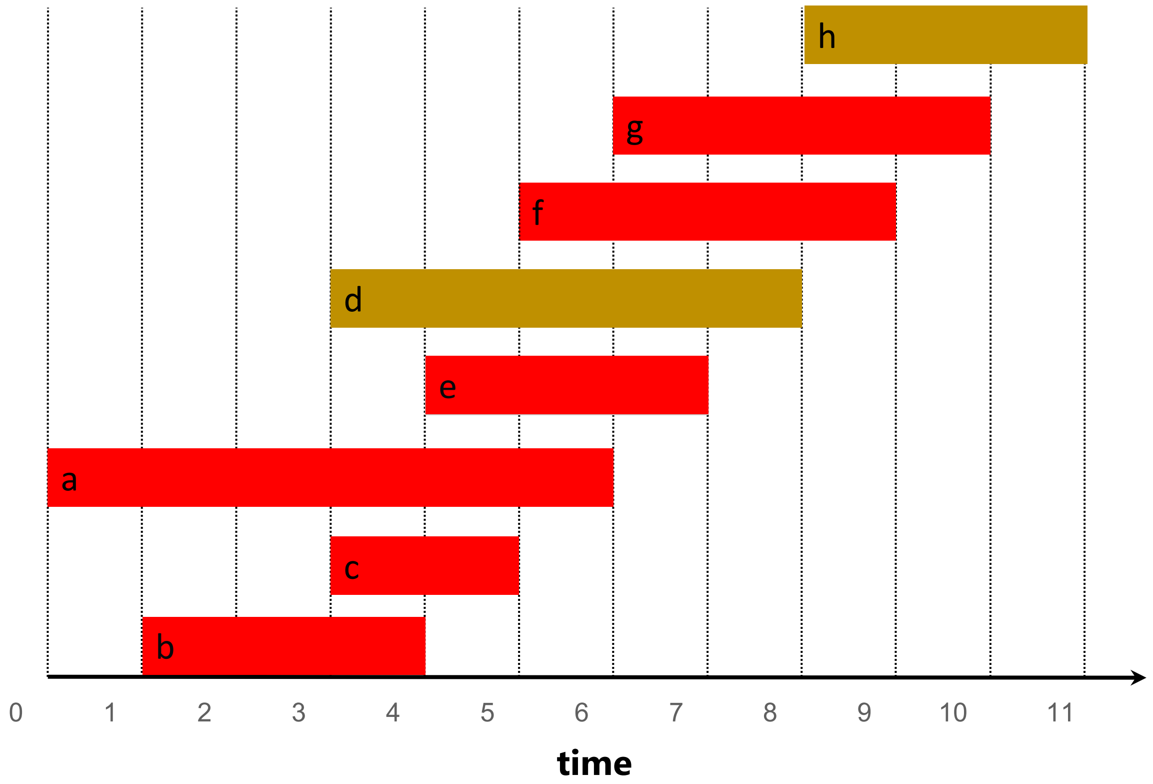

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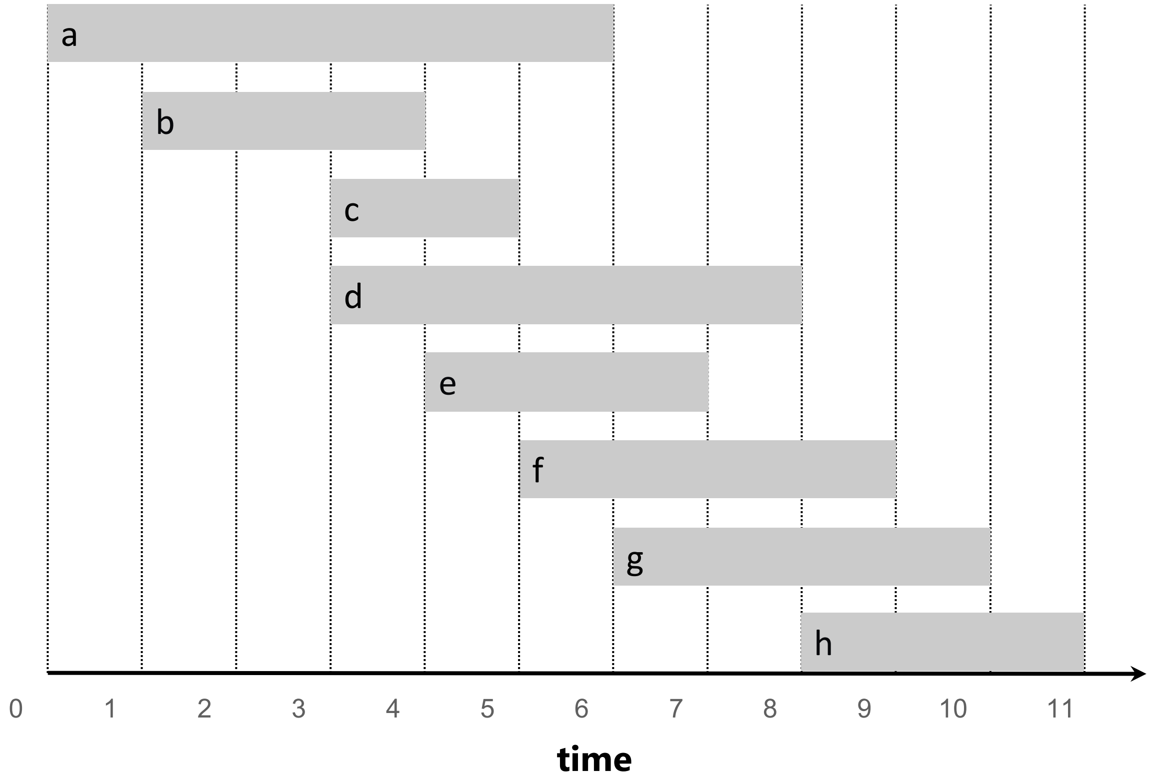

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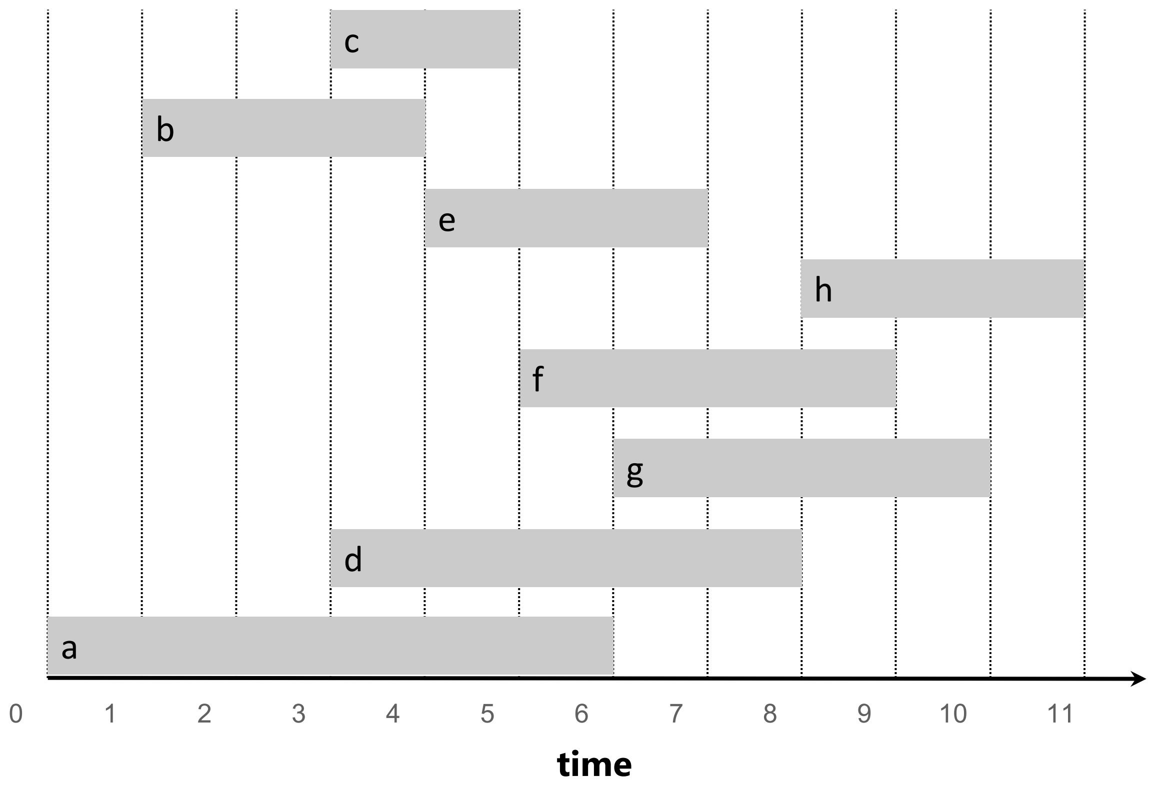

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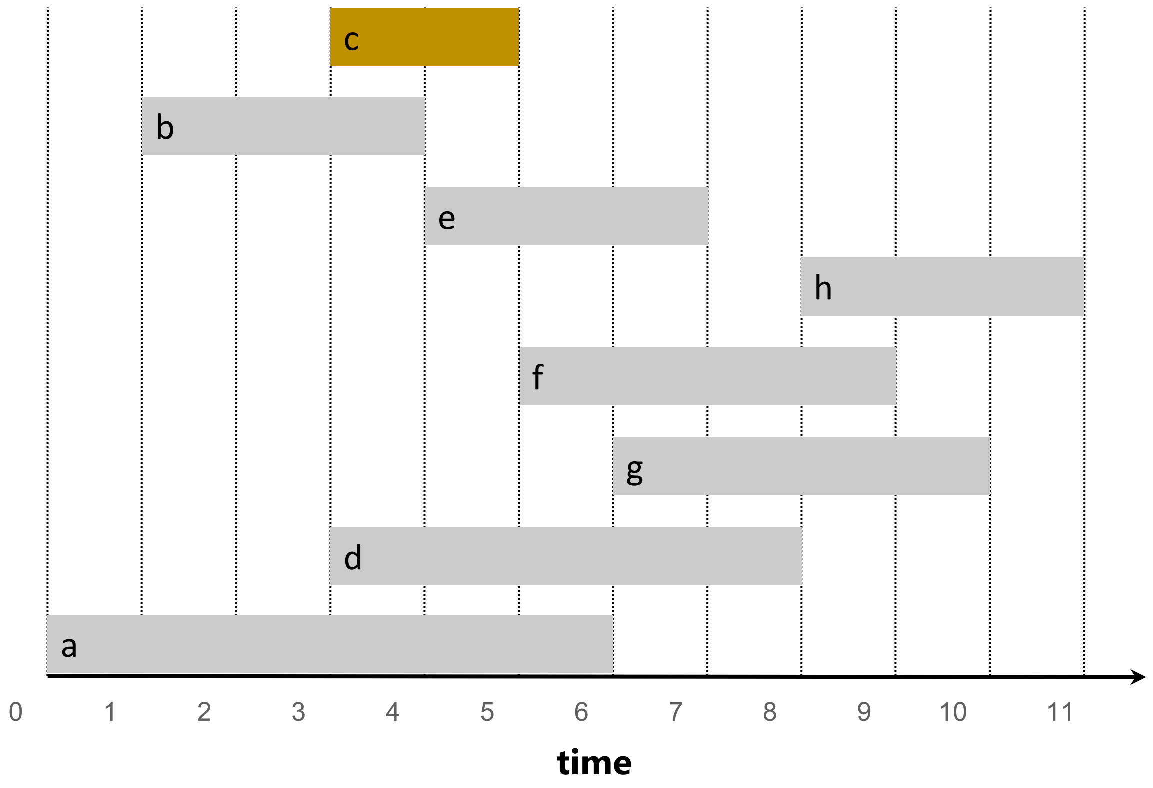

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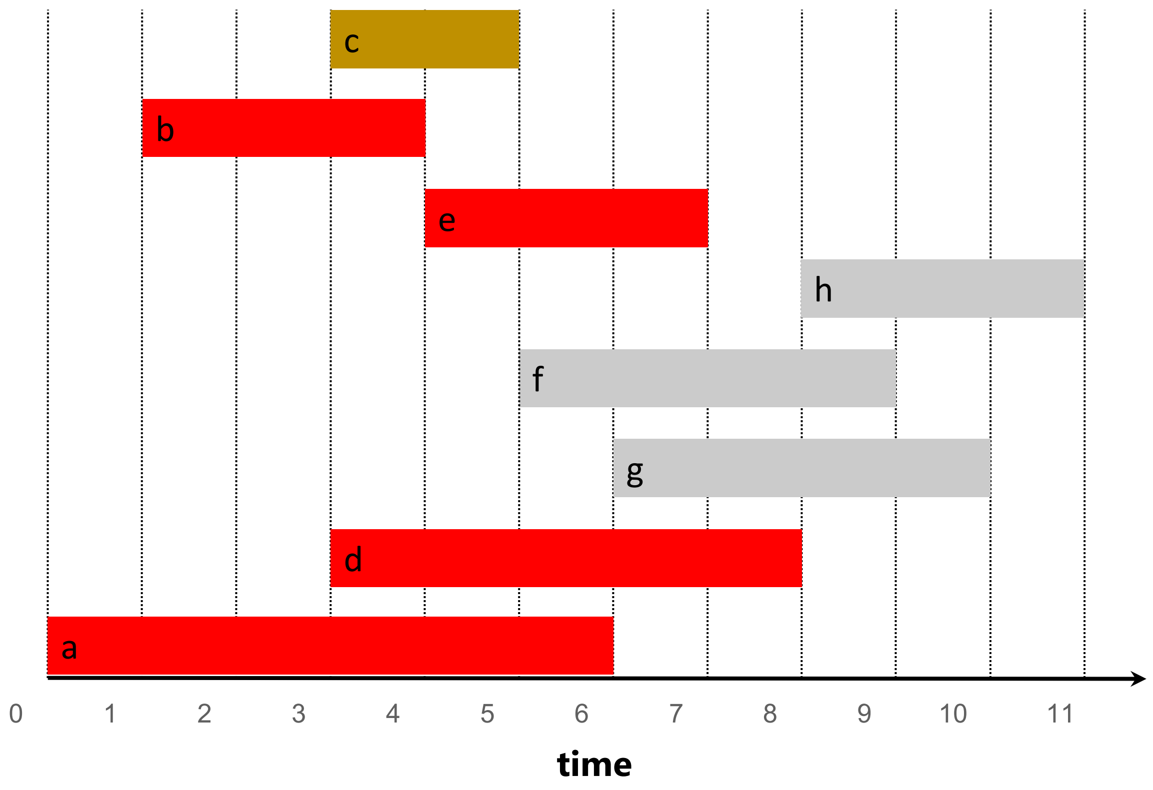

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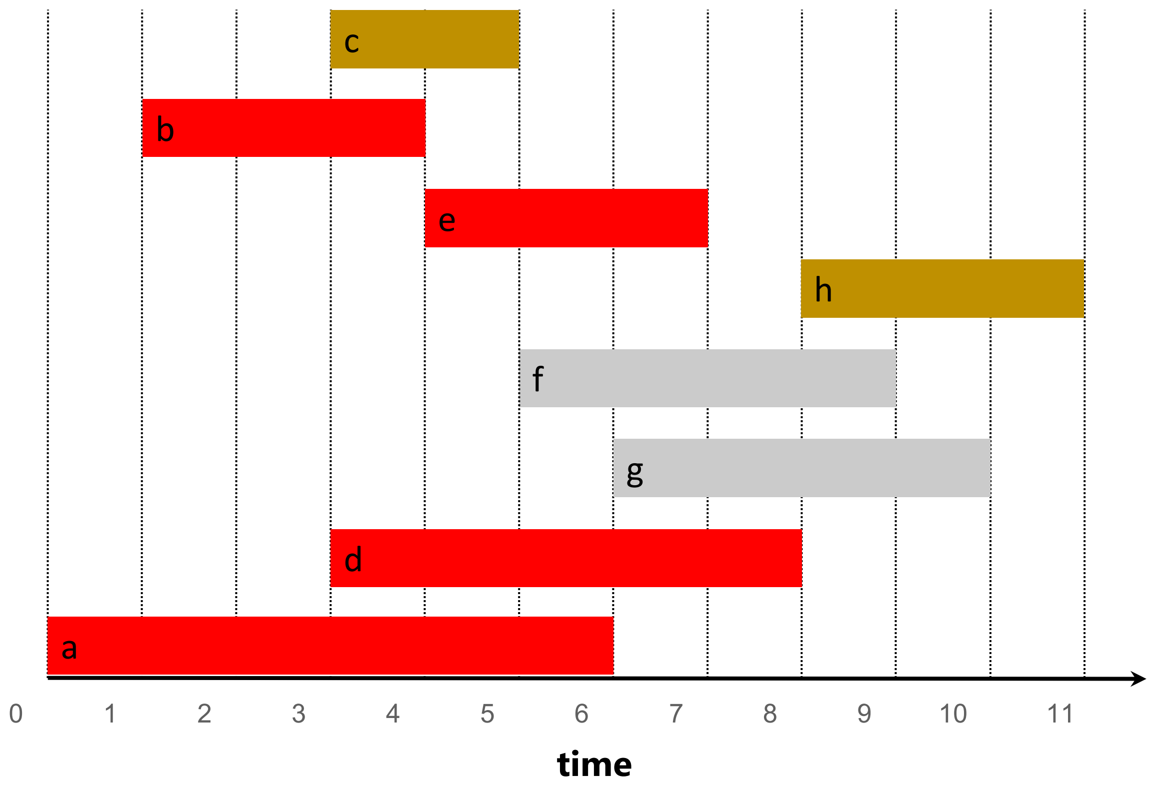

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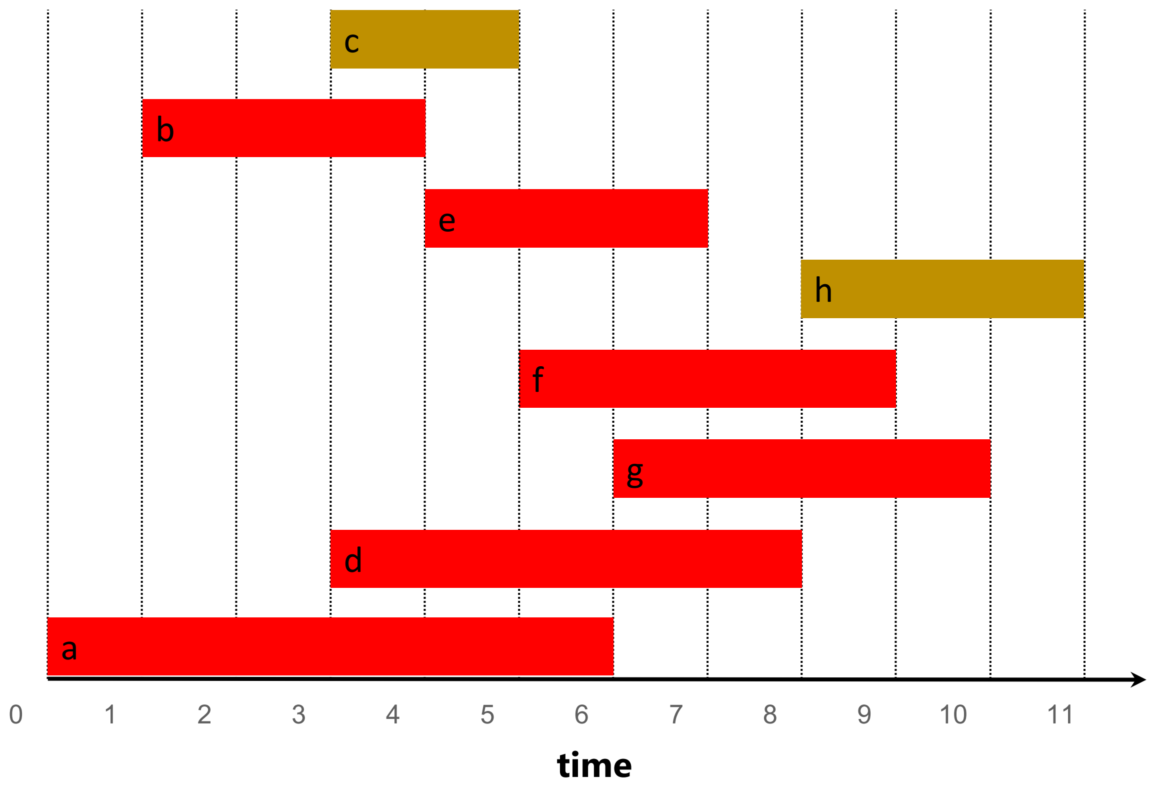

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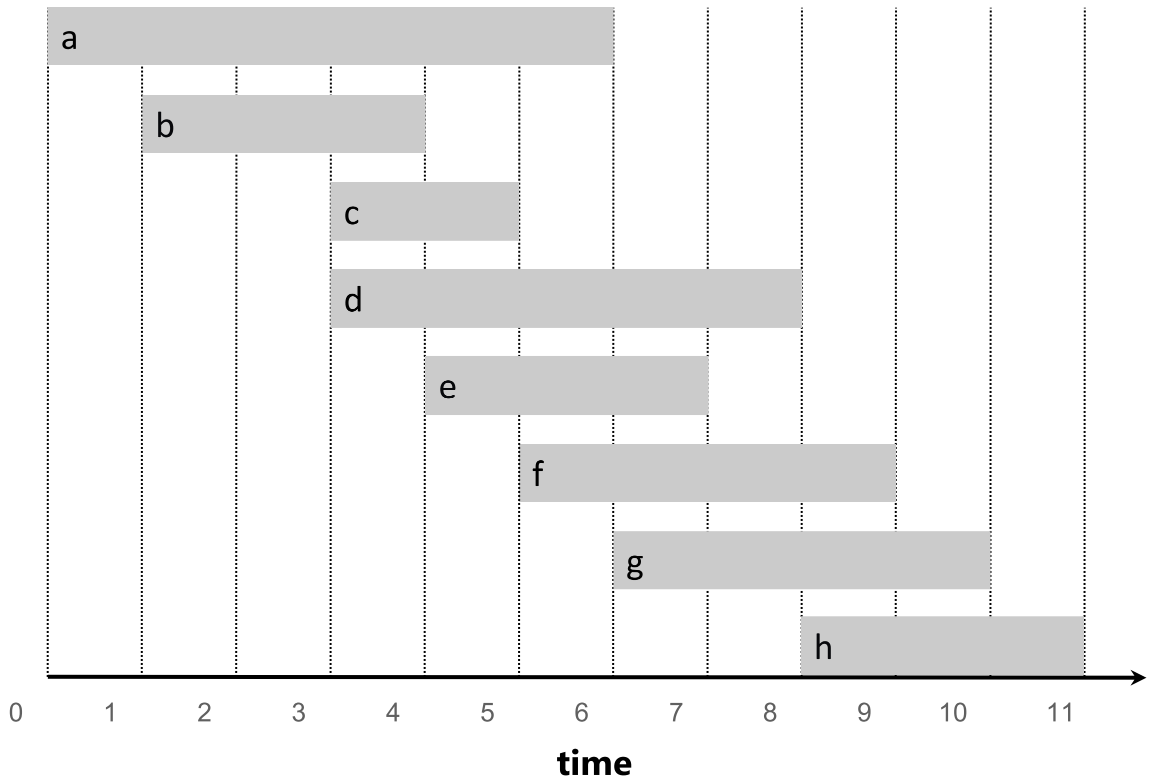

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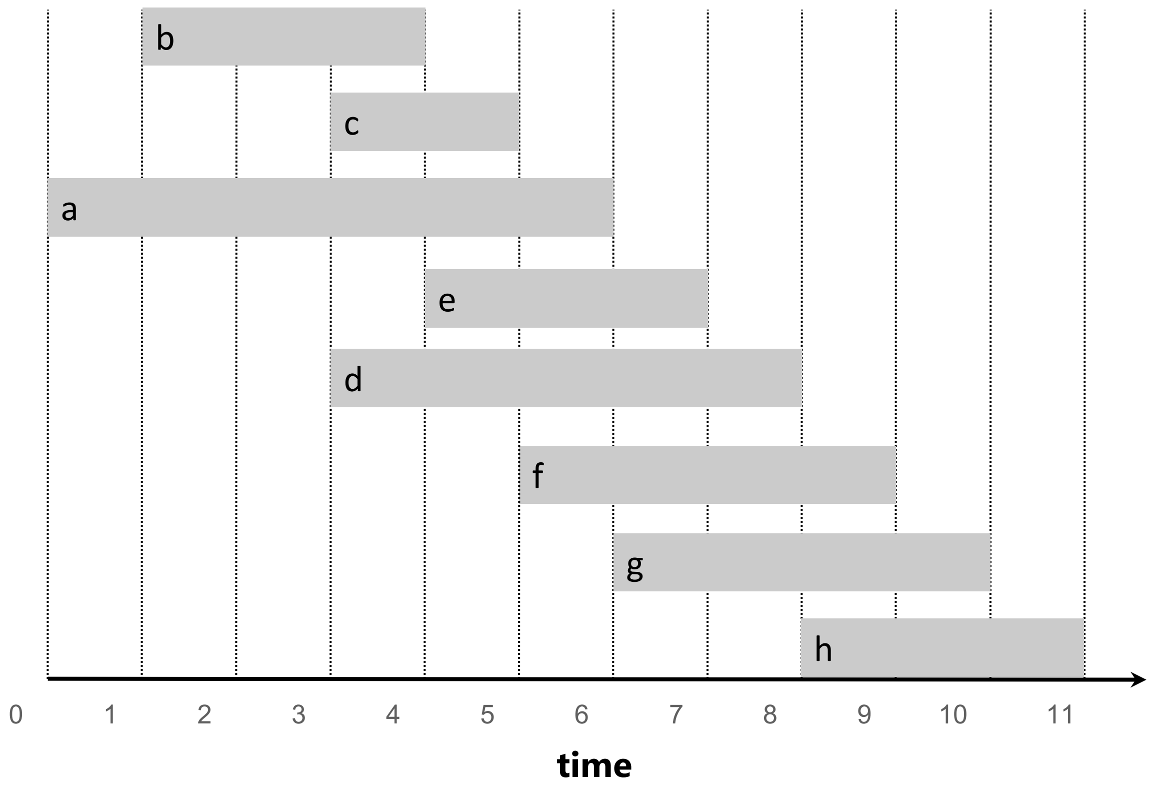

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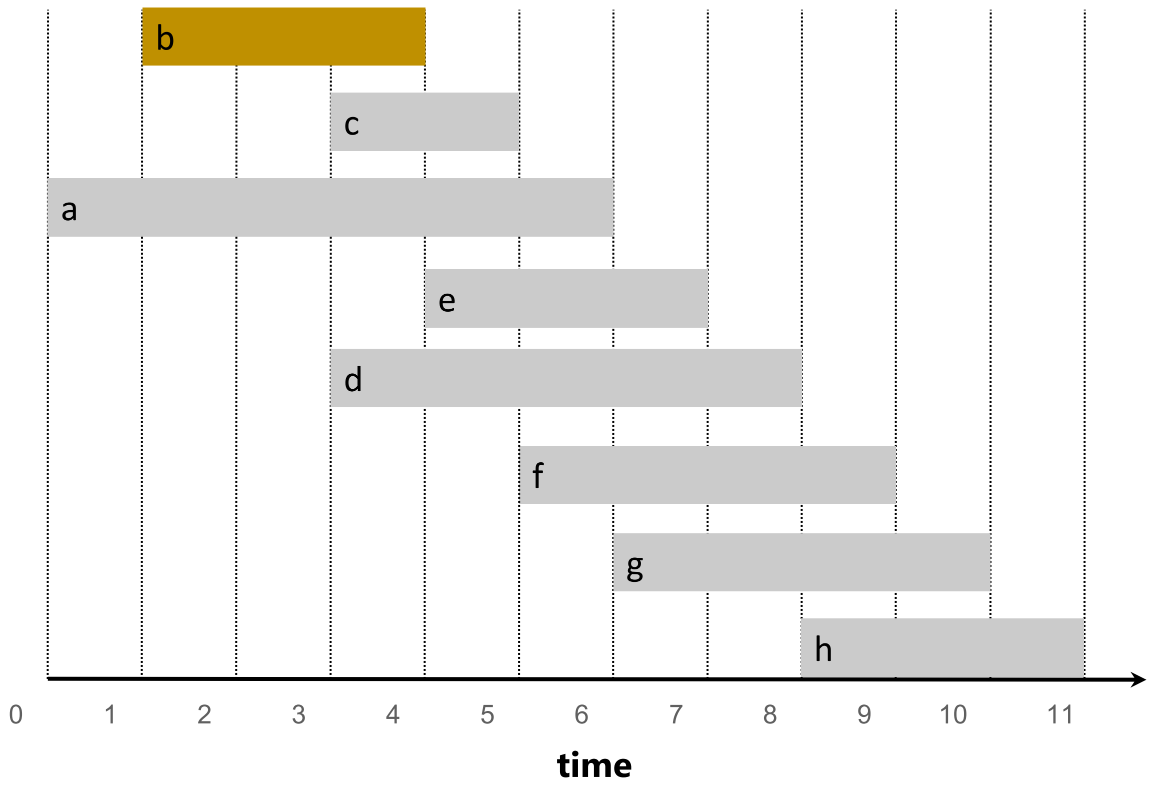

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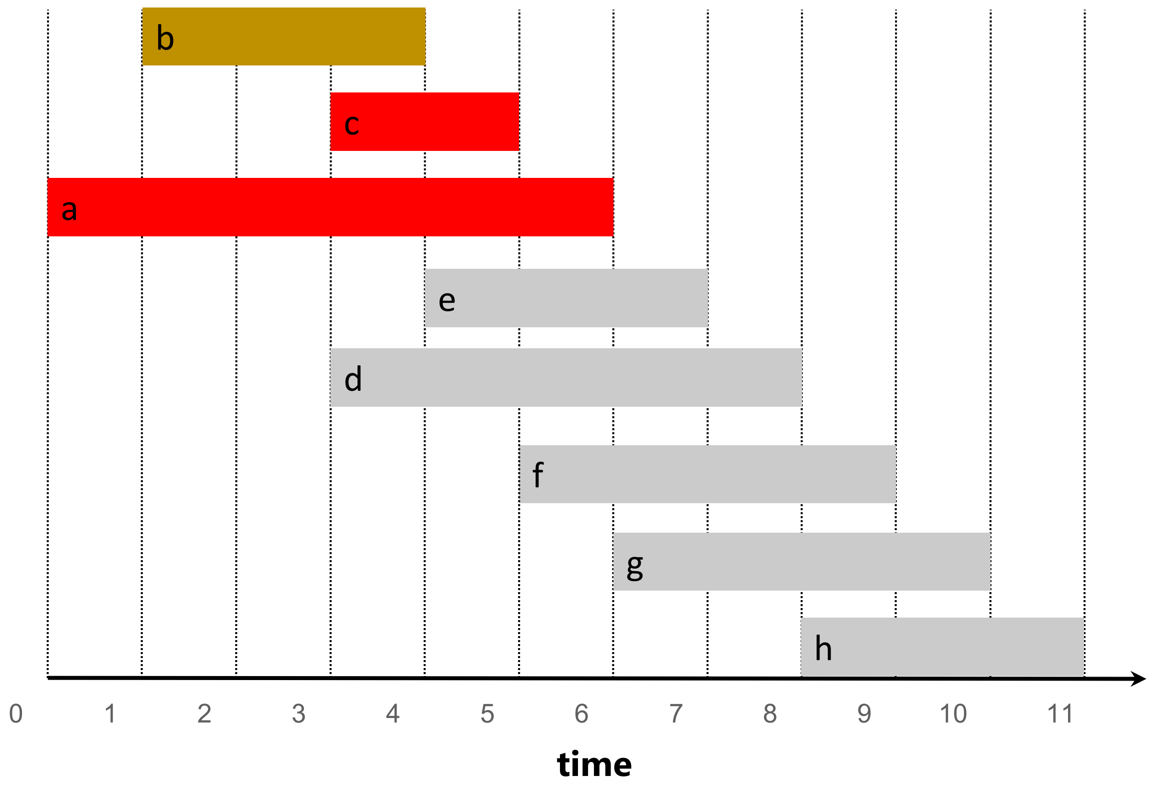

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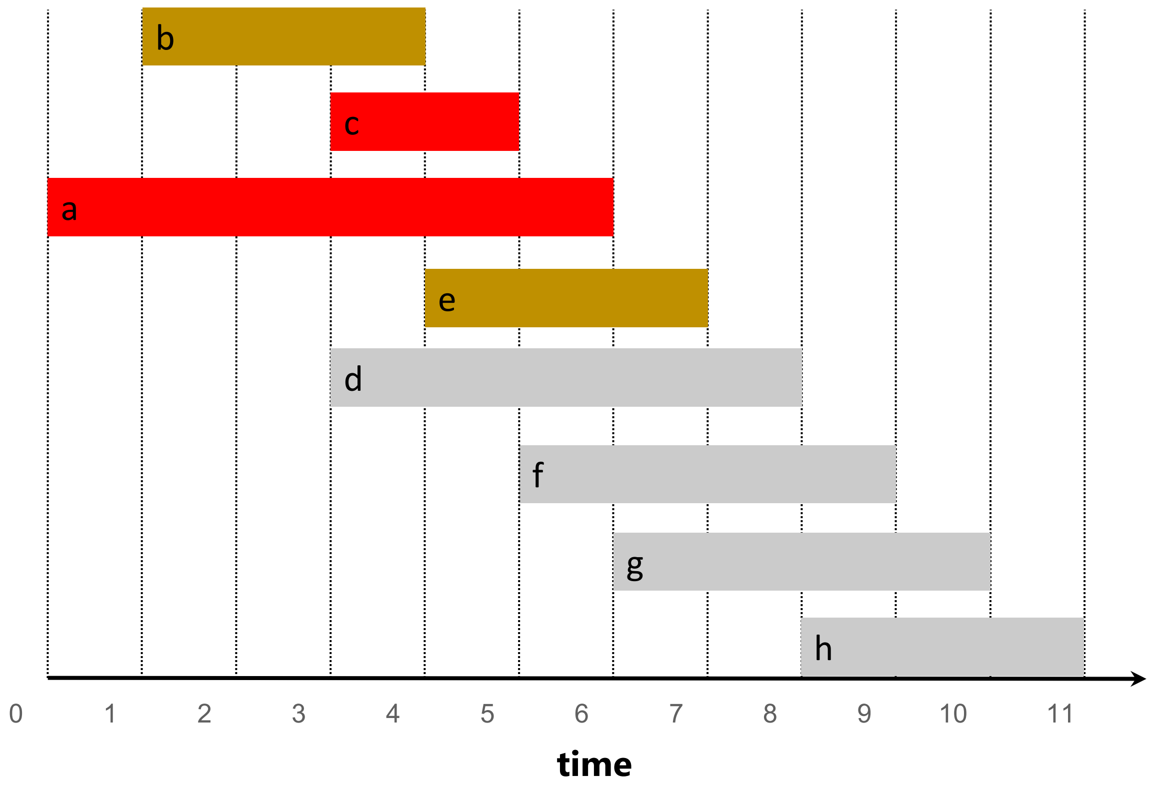

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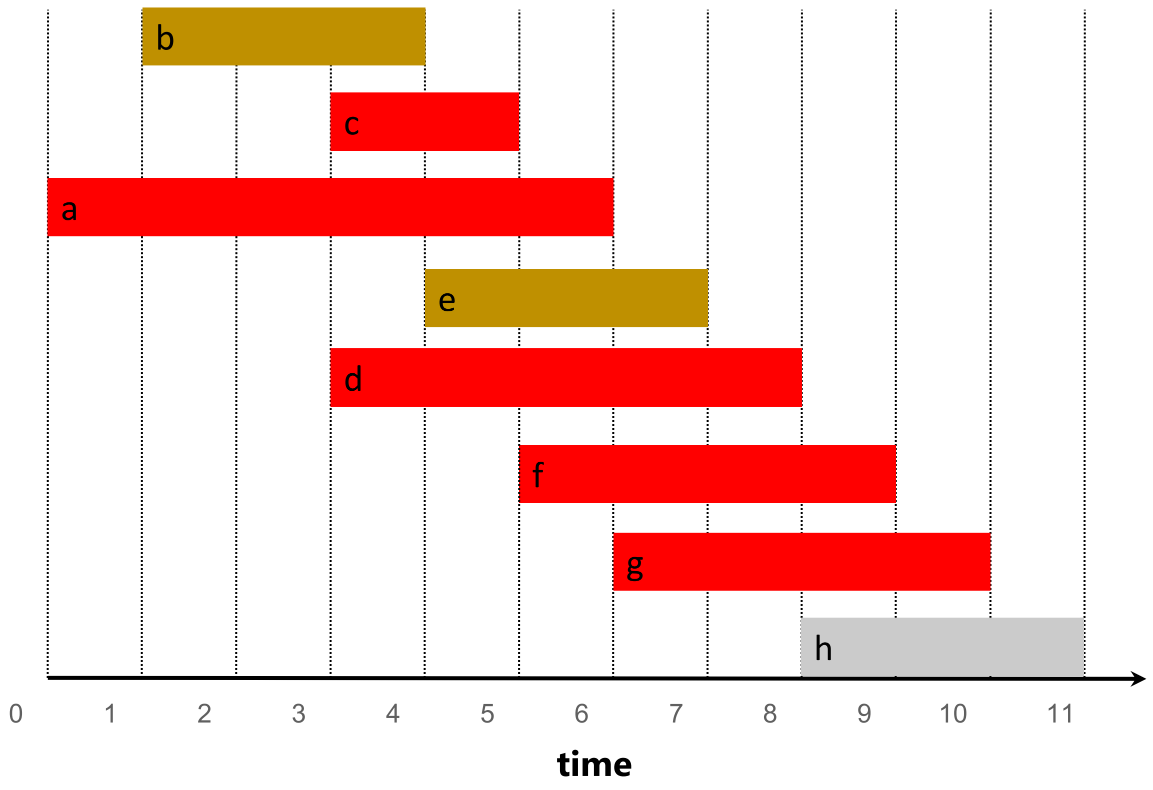

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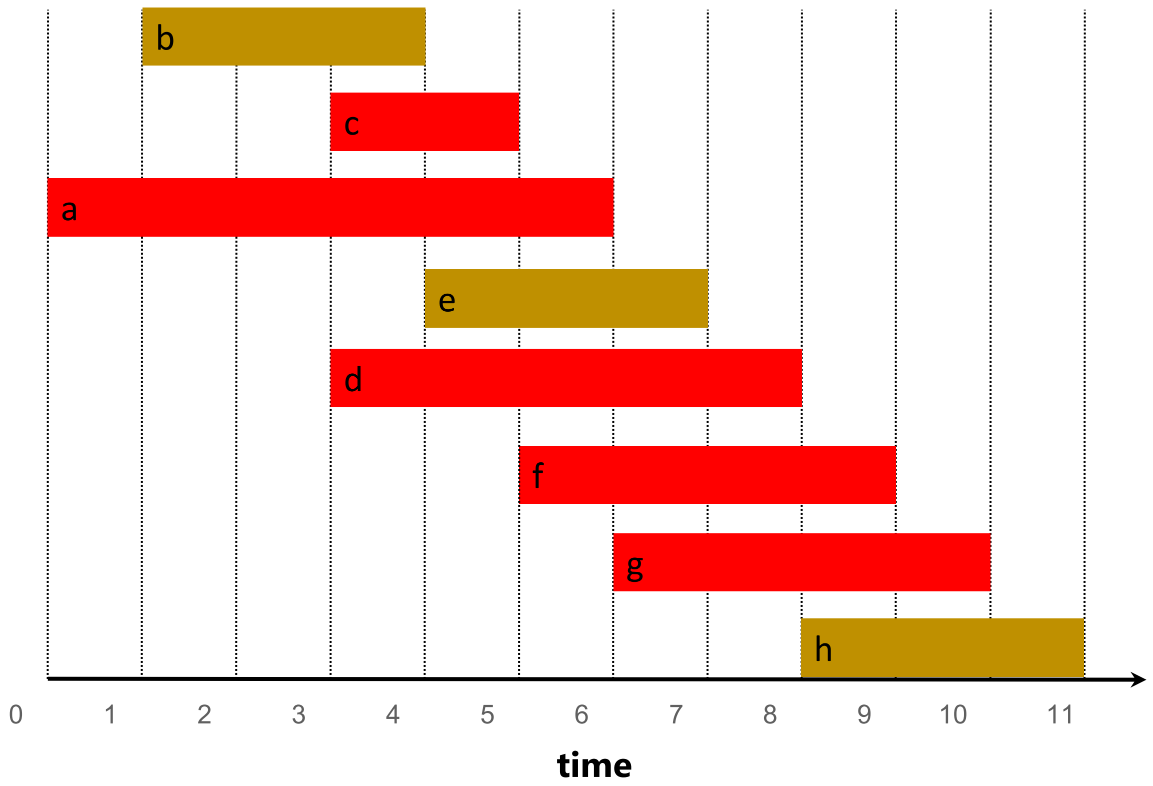

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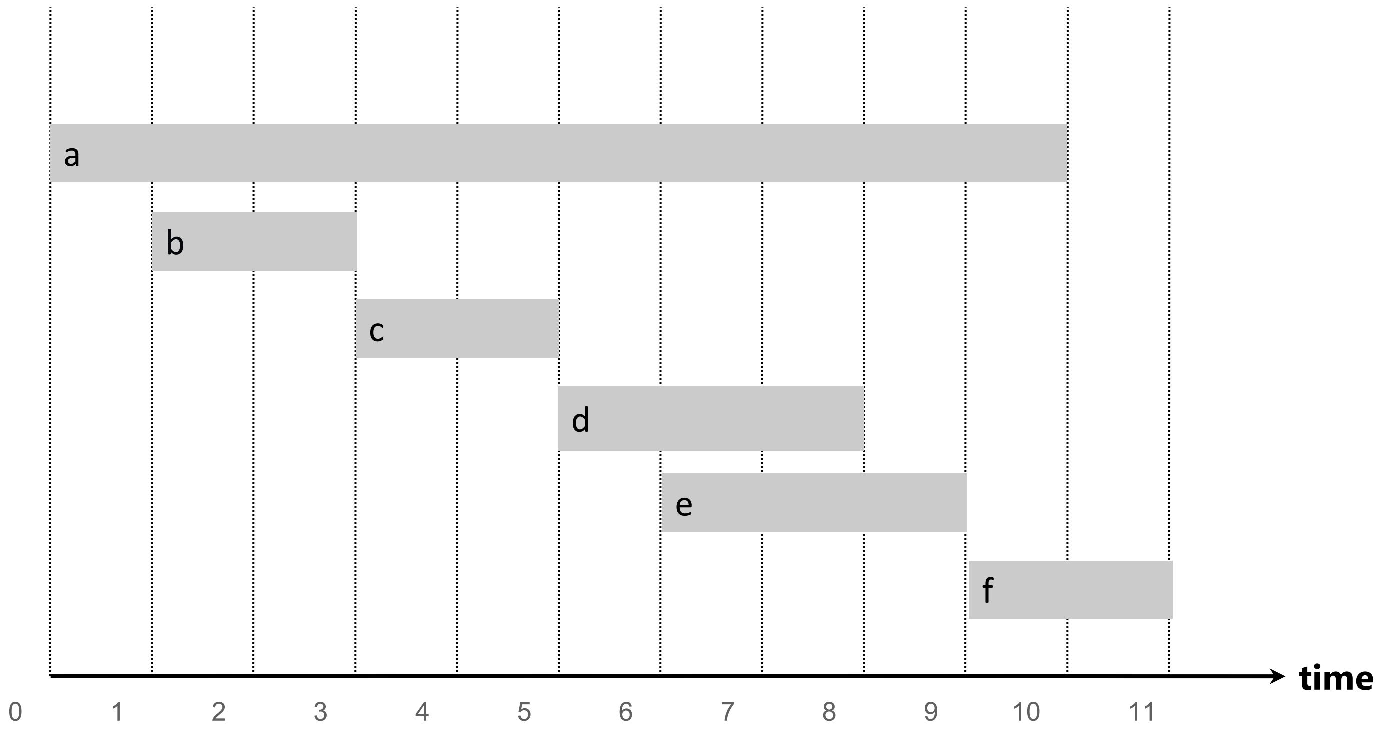

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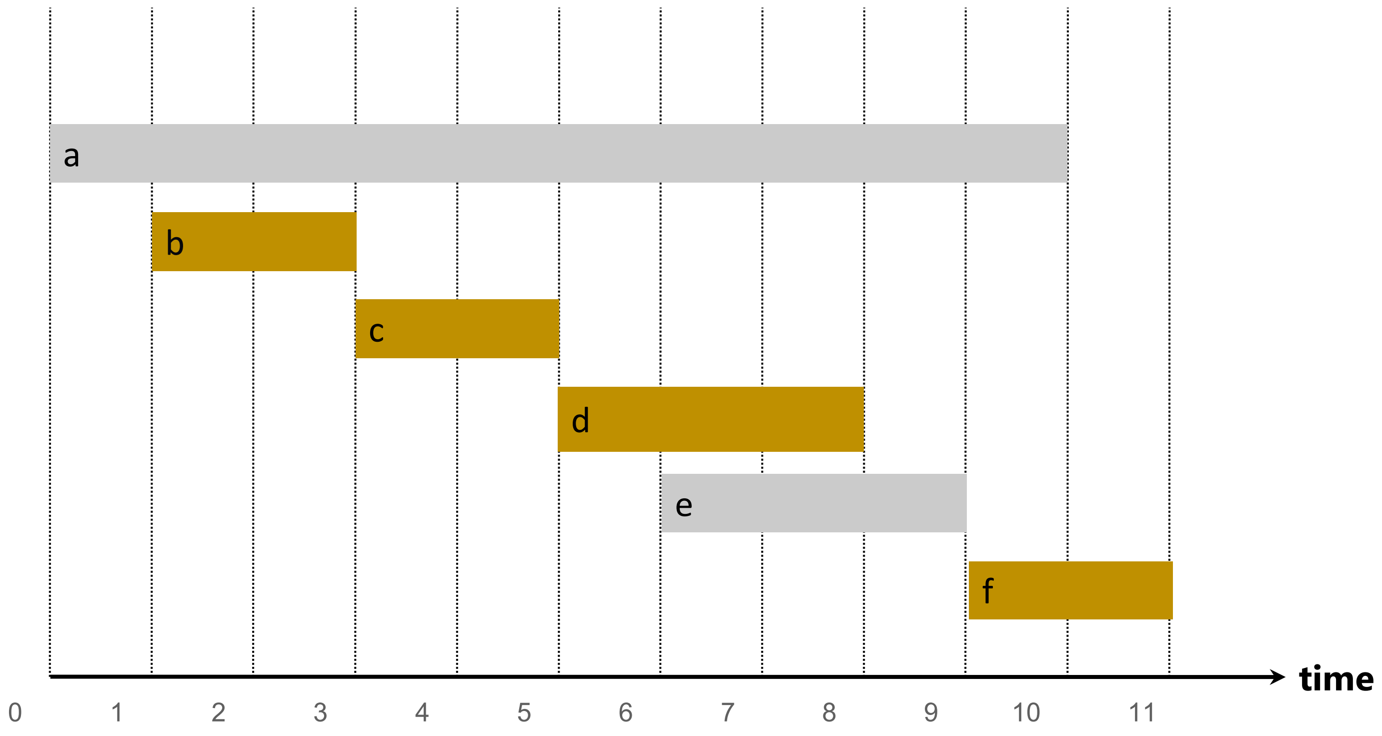


Interval Scheduling: EFTF Algorithm Analysis

EARLIEST-FINISH-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT jobs by finish times and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

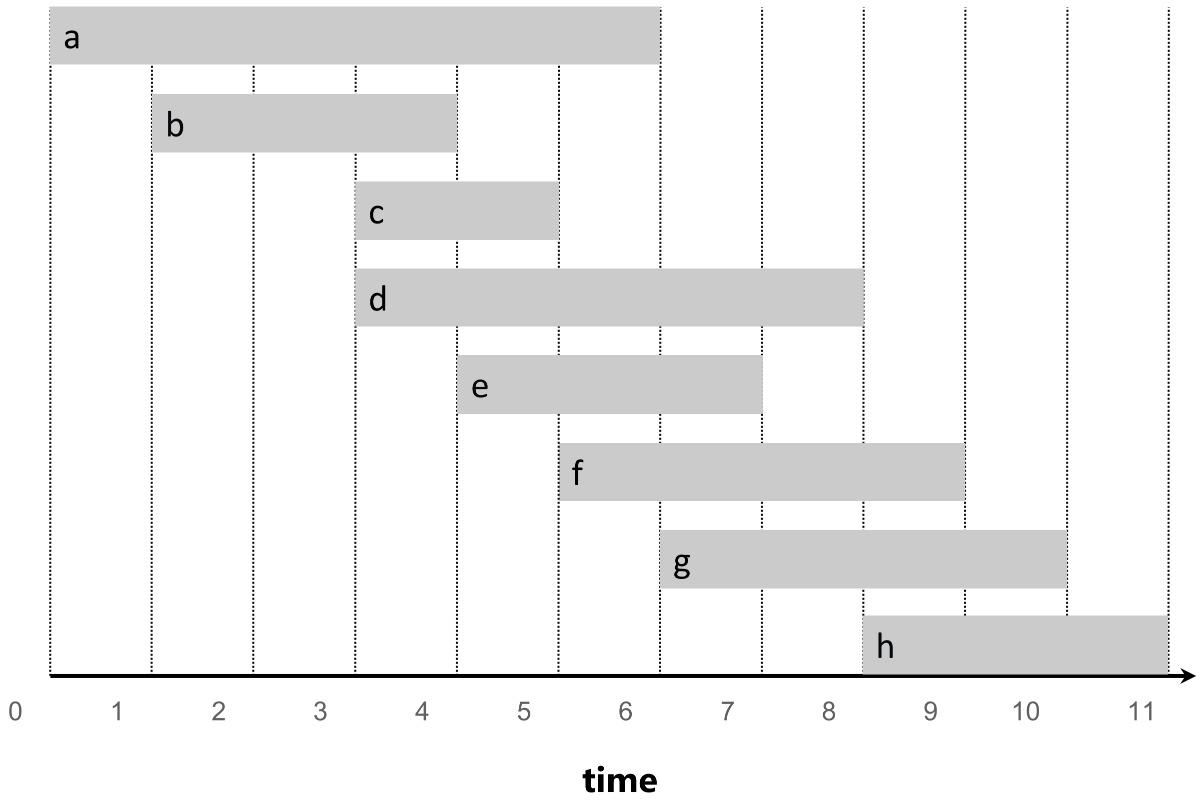
$S \leftarrow \varnothing$. ⟵ set of jobs selected

FOR $j = 1$ TO n

IF (*job j is compatible with S*)

$S \leftarrow S \cup \{ j \}$.

RETURN S.



Interval Scheduling: EFTF Algorithm Analysis

EARLIEST-FINISH-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT jobs by finish times and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

$Set \leftarrow \varnothing$. set of jobs selected

FOR $j = 1$ TO n

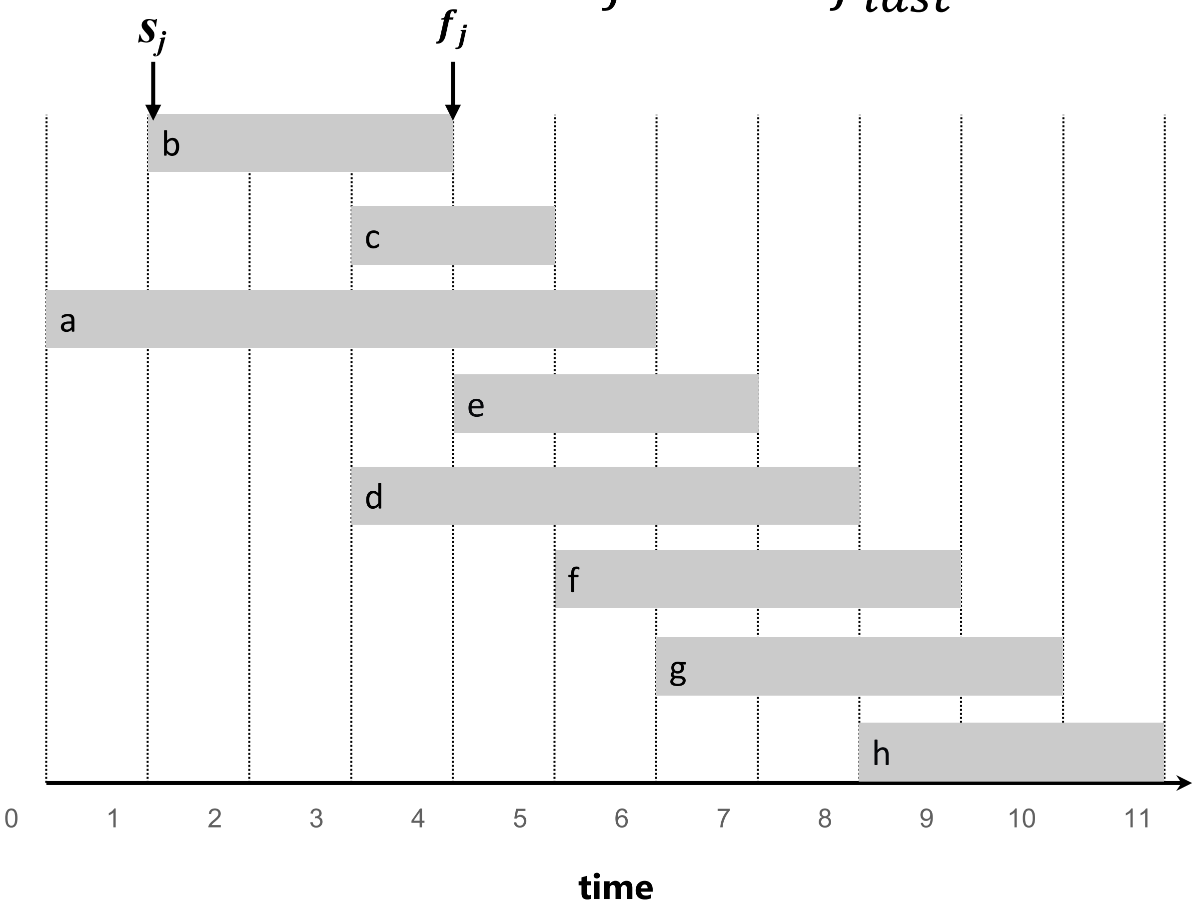
IF (job j is compatible with Set)

$Set \leftarrow Set \cup \{ j \}$.

RETURN S.

$$\boldsymbol{\Theta(n \log n) + \Theta(n)}$$

$$s_j \geq Set_{f_{last}}$$



Interval Scheduling: Correctness

Theorem: The earliest-finish-time-first (EFTF) algorithm is optimal.

Let A= $i_1, i_2, i_3, i_4, \ldots i_k$ ⟵ **EFTF**

Assume sorted: f(i_1) ≤ f(i_2) ≤ f(i_3) …

Let B= $j_1, j_2, j_3, j_4, \ldots j_k, j_{k+1}, \ldots j_m,$ ⟵ **Magic Optimal**

Assume sorted: f(j_1) ≤ f(j_2) ≤ f(j_3) …

where **m** > **k**

We need to show m ≤ k

Interval Scheduling: Correctness

Let A= $i_1, i_2, i_3, i_4, \dots i_k$

Assume sorted: f(i_1) ≤ f(i_2) ≤ f(i_3) …

Let B= $j_1, j_2, j_3, j_4, \dots j_k, j_{k+1}, \dots j_m,$

Assume sorted: f(j_1) ≤ f(j_2) ≤ f(j_3) …

First, we need to show that for each **r ≤ k**, f(i_r) ≥ f(j_r) [by Induction]

Then, we need to show that for each **m ≤ k**, [by Contradiction]

Interval Scheduling: Correctness

Proof: [by Induction]

Base Case:

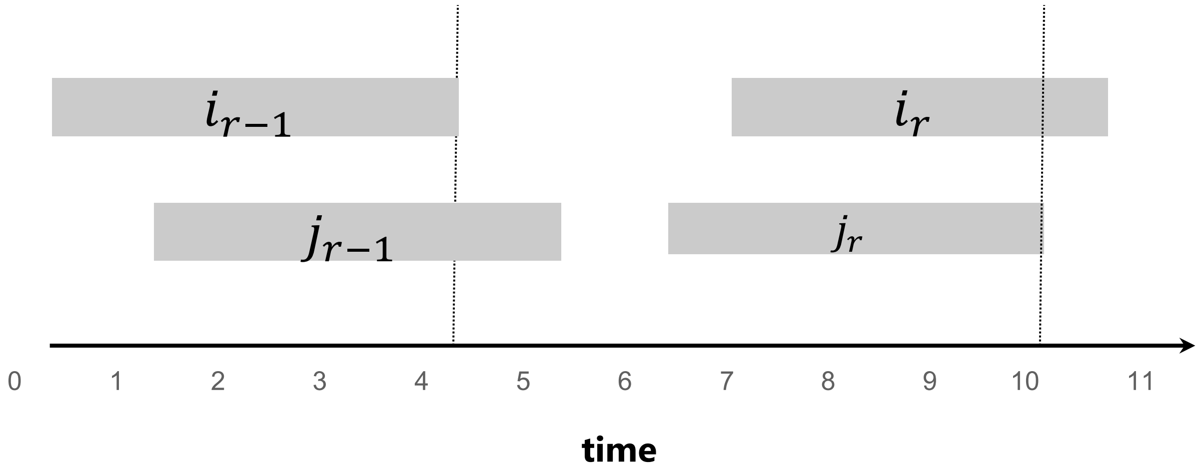
- r=1: EFTF chooses booking i_1 with earliest overall finish time, i.e., $\mathbf{f}(\boldsymbol{i_1}) \leq \mathbf{f}(\boldsymbol{j_1})$

Inductive Hypothesis:

- r> 1: Assume, by induction that $\mathbf{f}(\boldsymbol{i_{r-1}}) \leq \mathbf{f}(\boldsymbol{j_{r-1}})$

Then

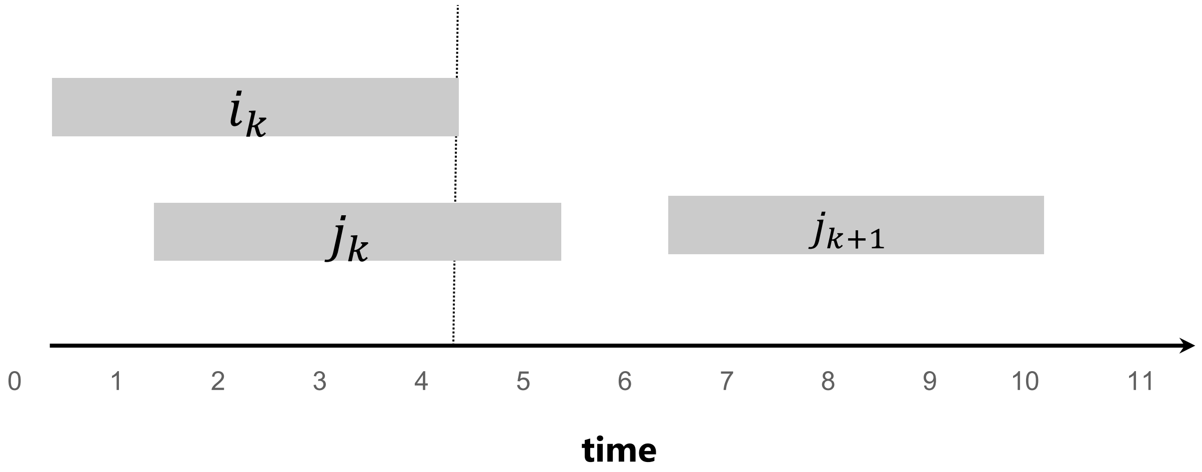
- It must be the case that $\mathbf{f}(\boldsymbol{i_r}) \leq \mathbf{f}(\boldsymbol{j_r})$



Interval Scheduling: Correctness

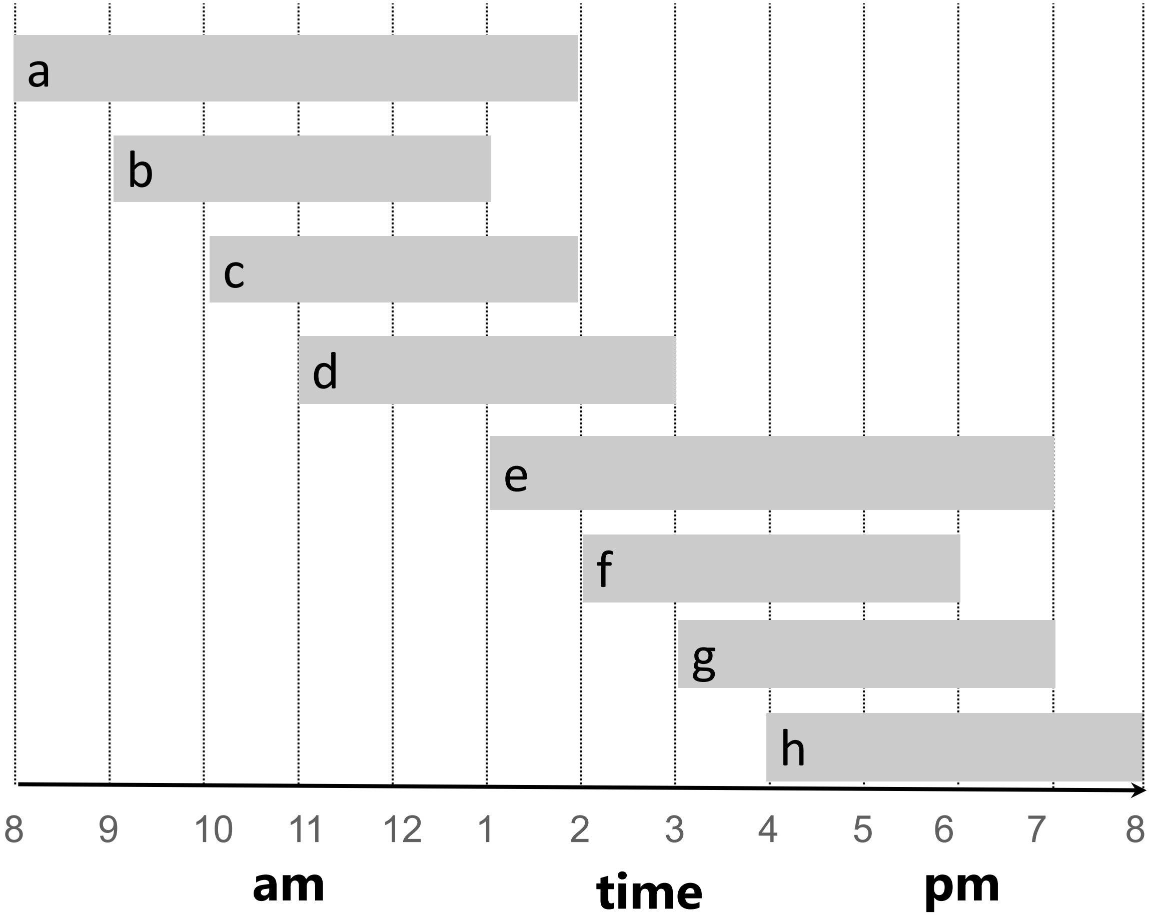
Proof: [by Contradiction]

- We know that $\mathbf{f}(\boldsymbol{i_r}) \leq \mathbf{f}(\boldsymbol{j_r})$
- Consider j_{k+1} in “Magic Optimal”
- Greedy algorithms terminates only when no more jobs are left or all remaining jobs overlaps
- **Contradiction**

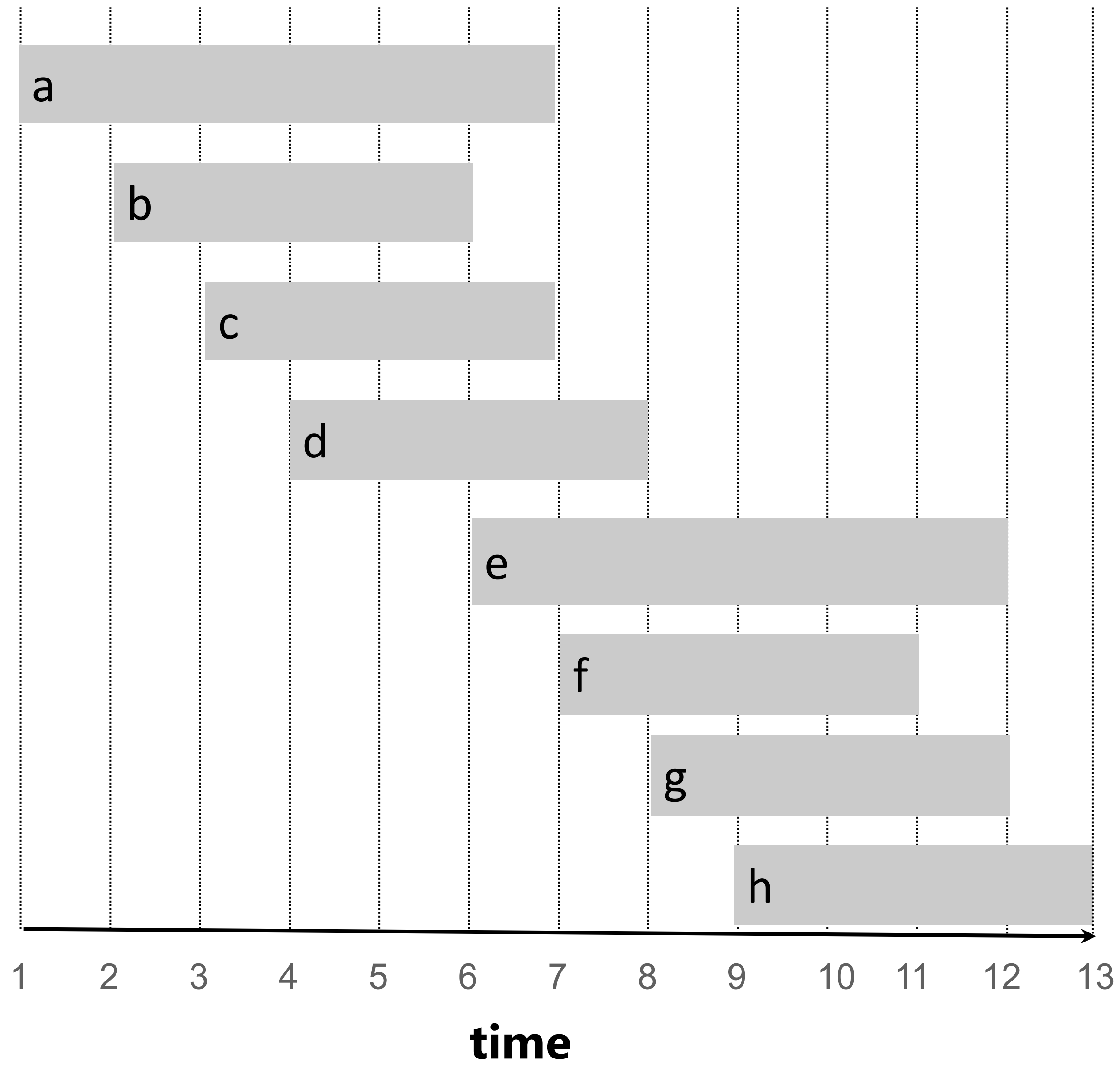


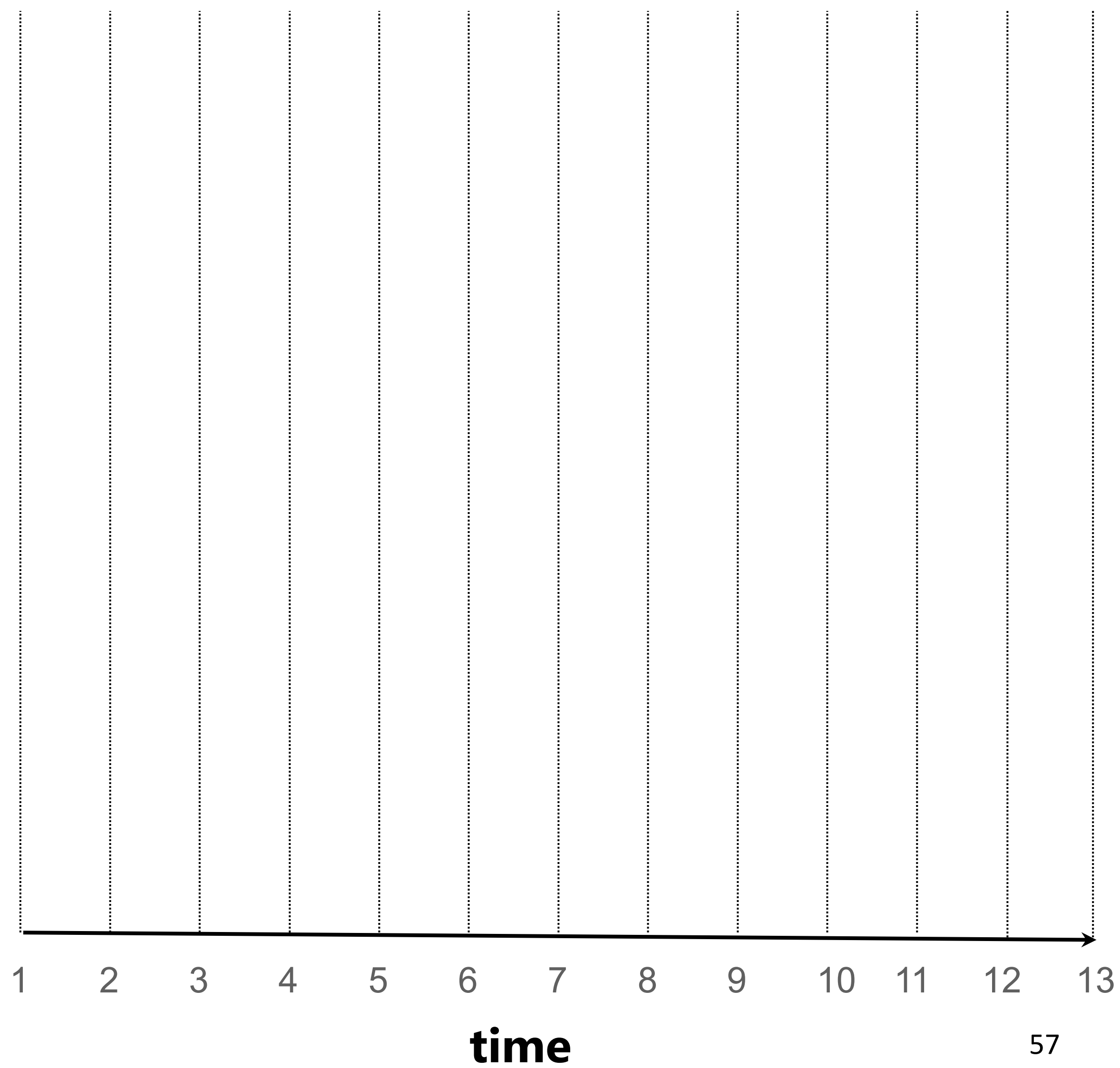
Interval partitioning

- Lecture j starts at s_j and finishes at f_j.
- **Goal:** find **minimum** number of classrooms to schedule all lectures so that no two lectures occur at the same time in the same room.



Earliest Finish Time First





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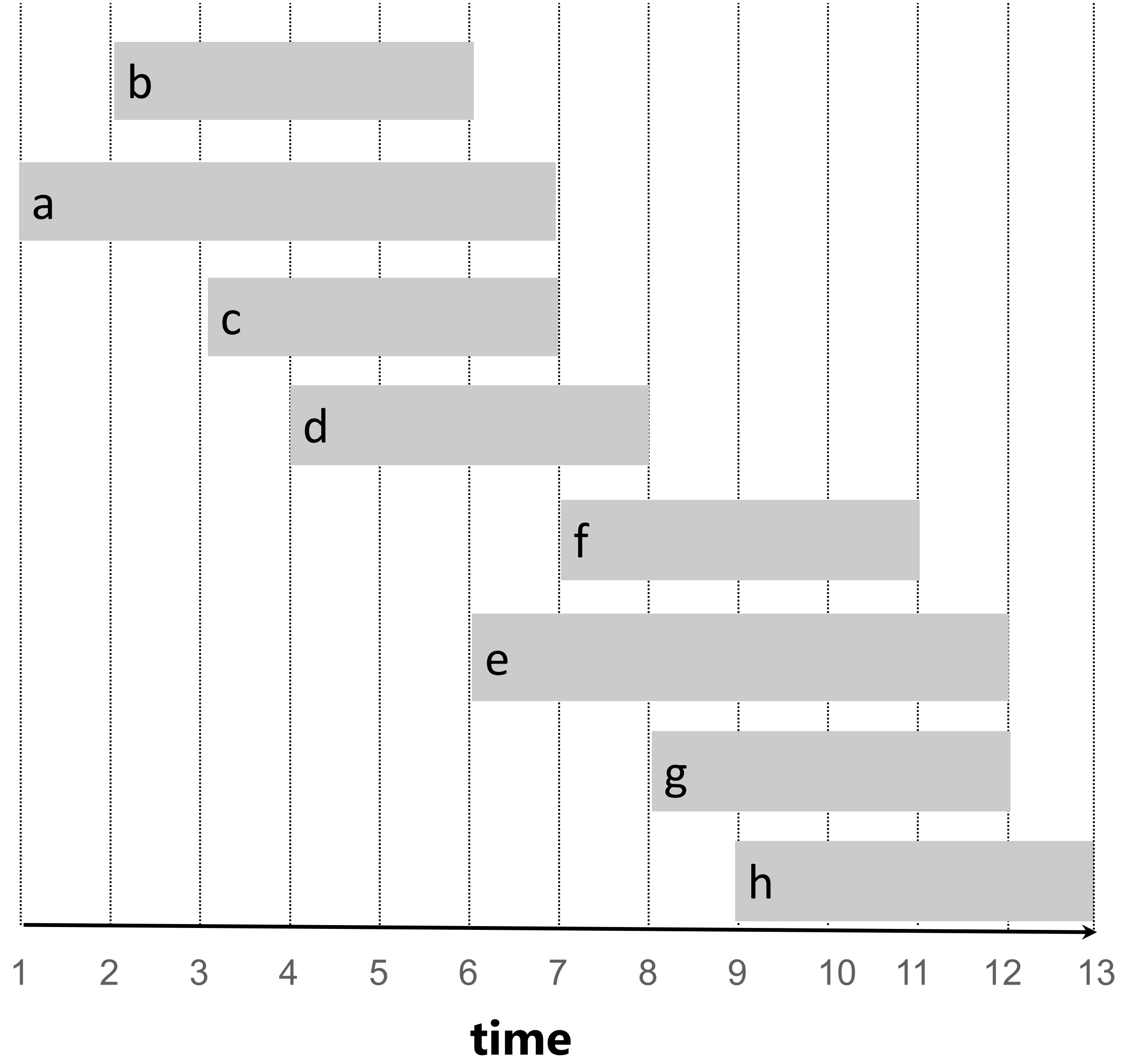
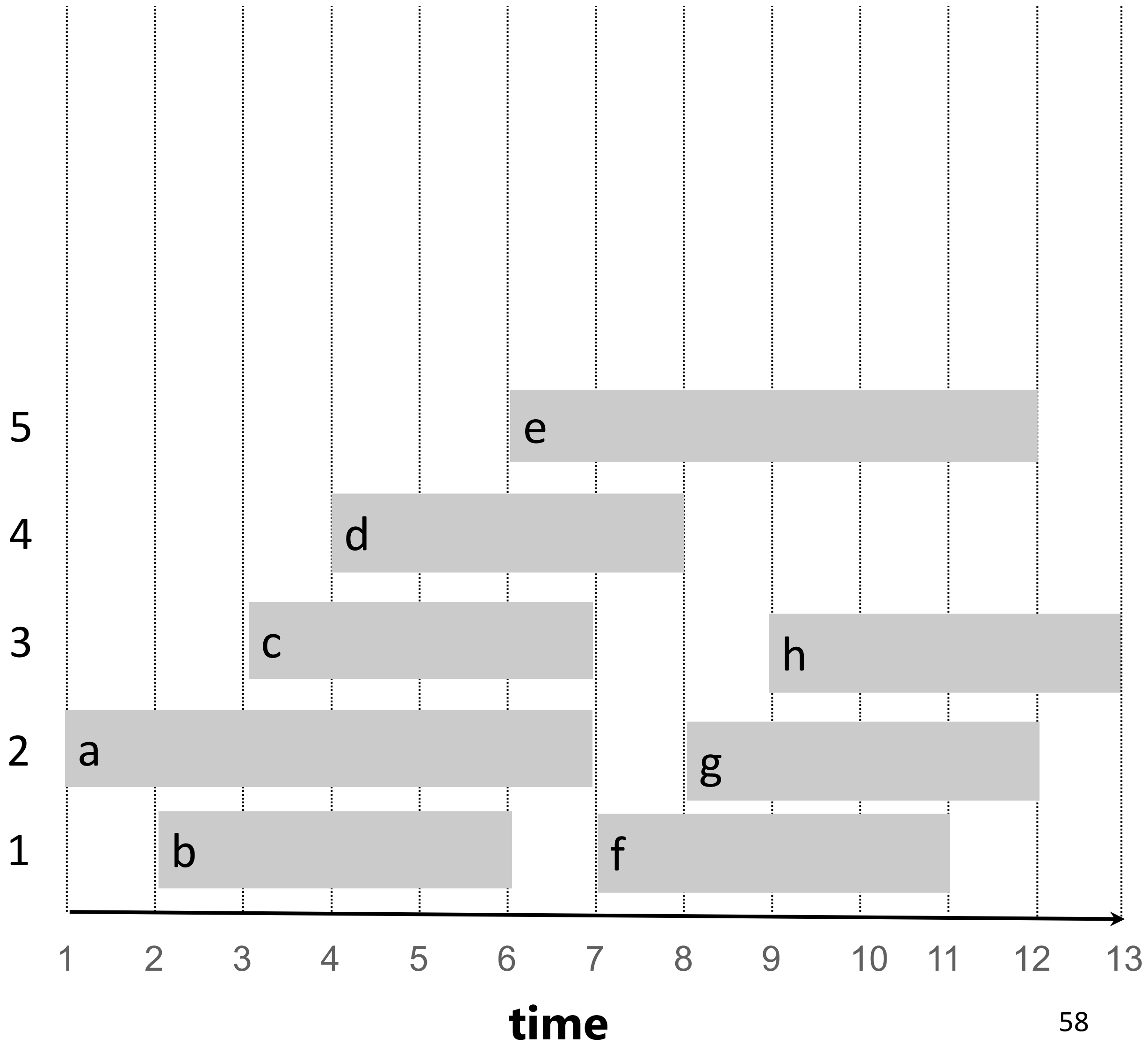
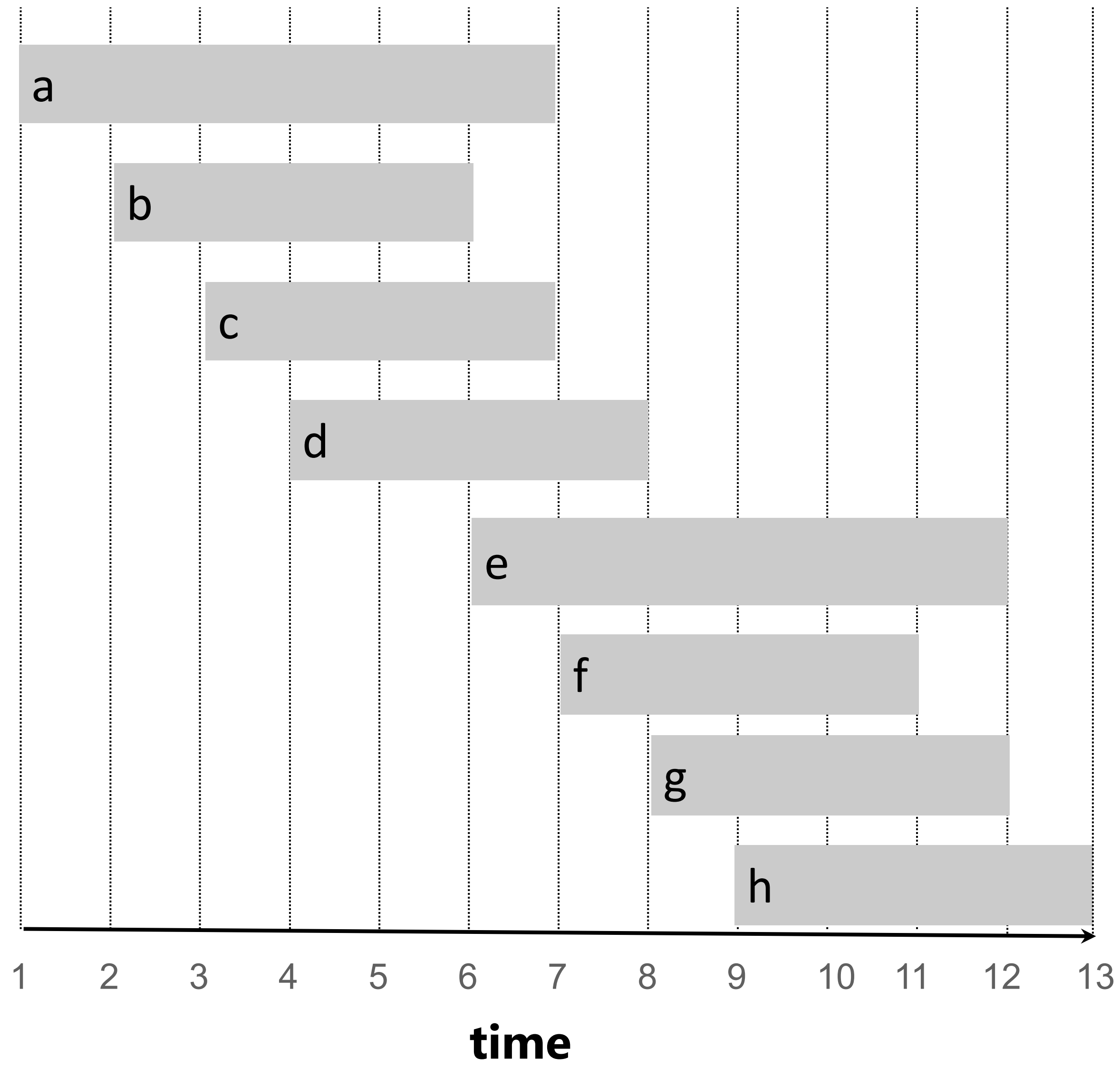
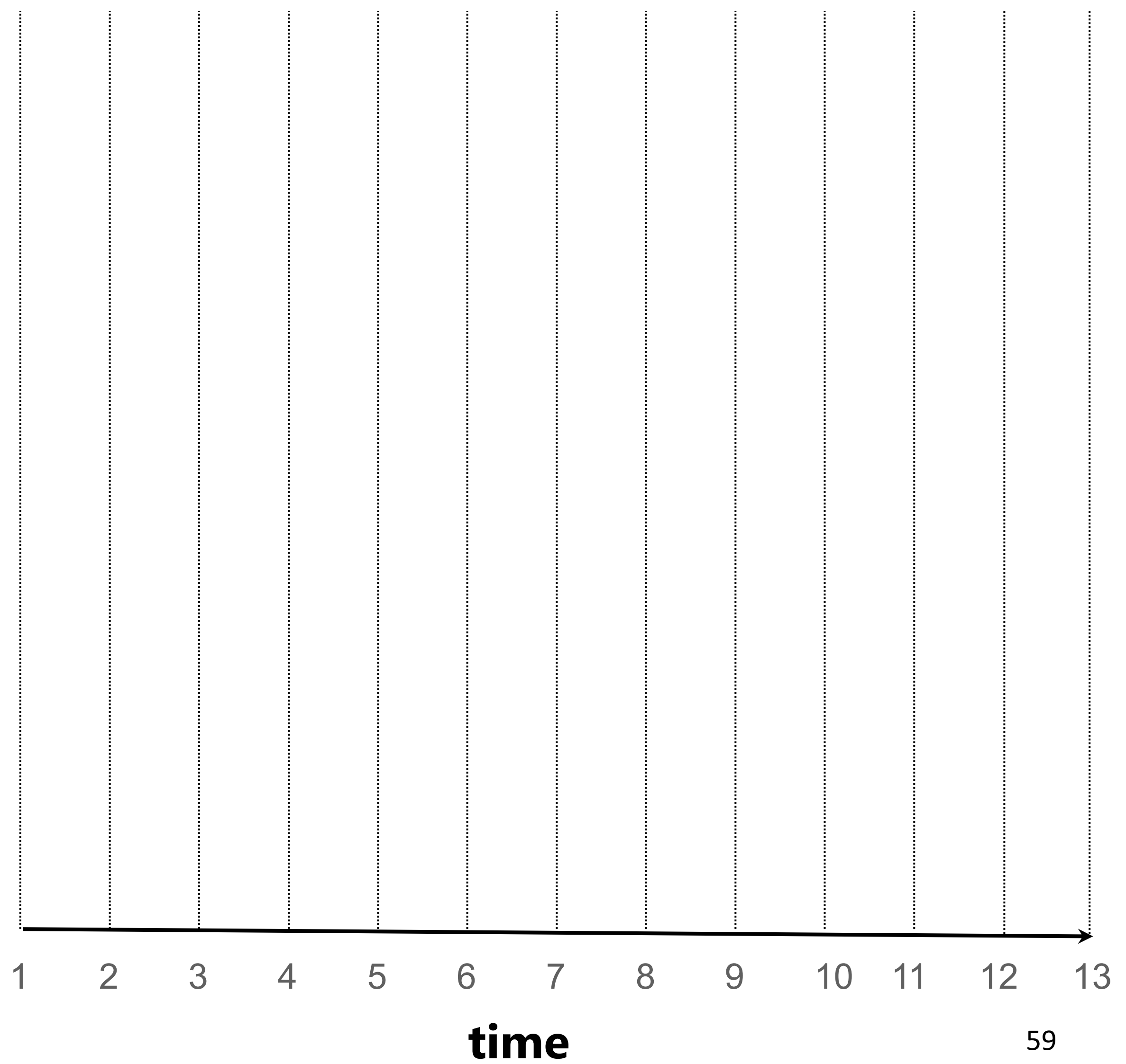


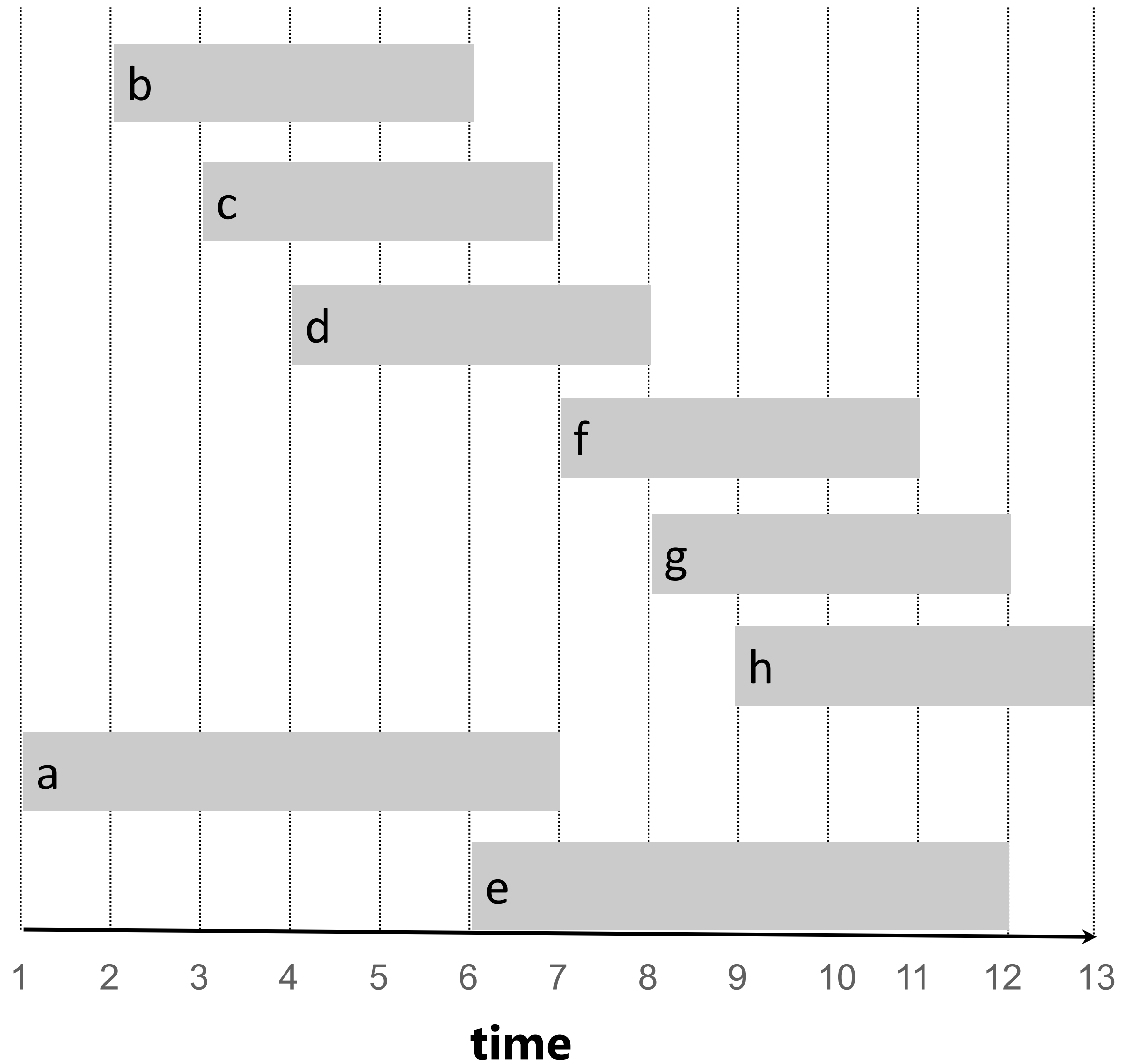


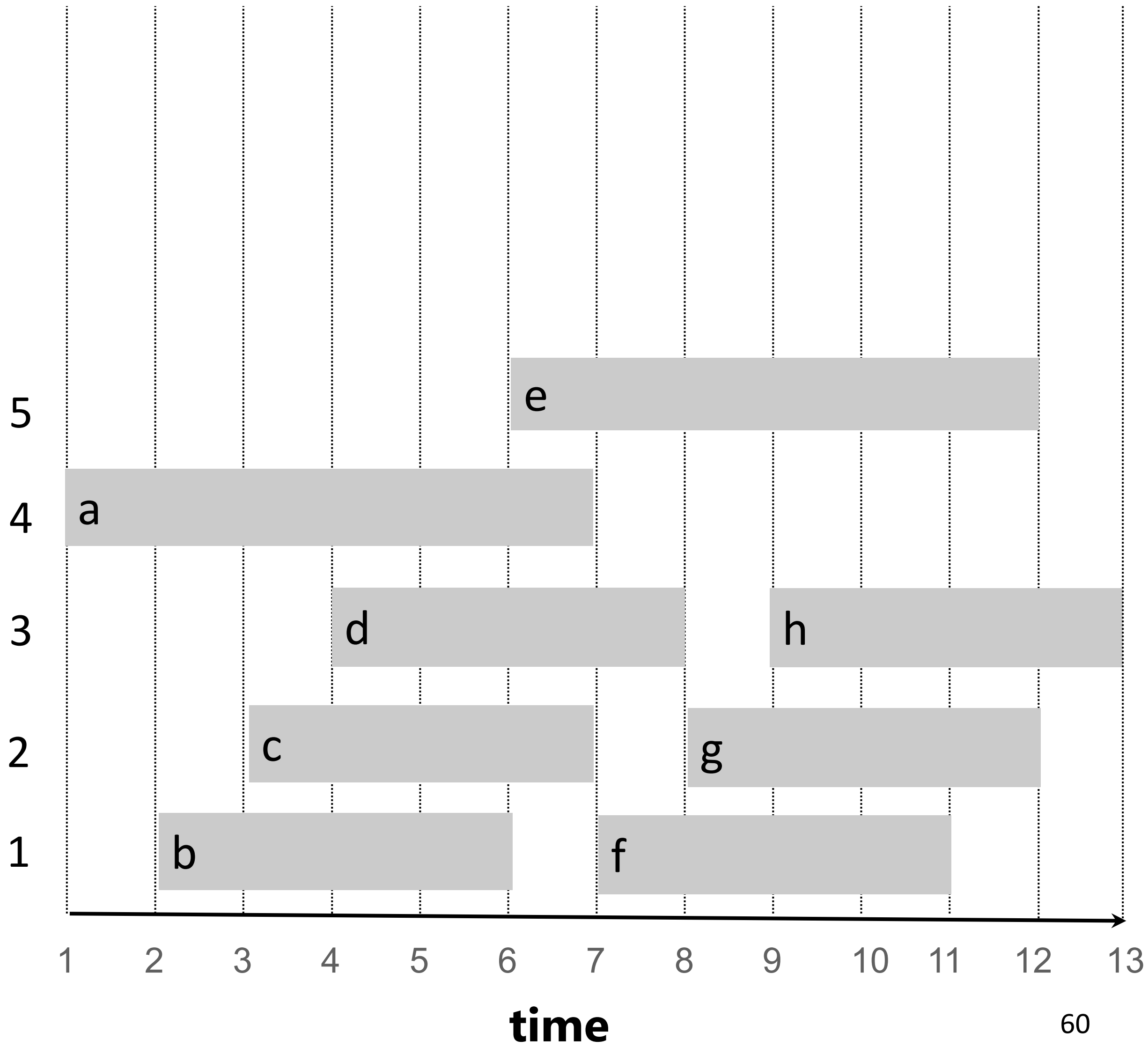
Smallest Interval First



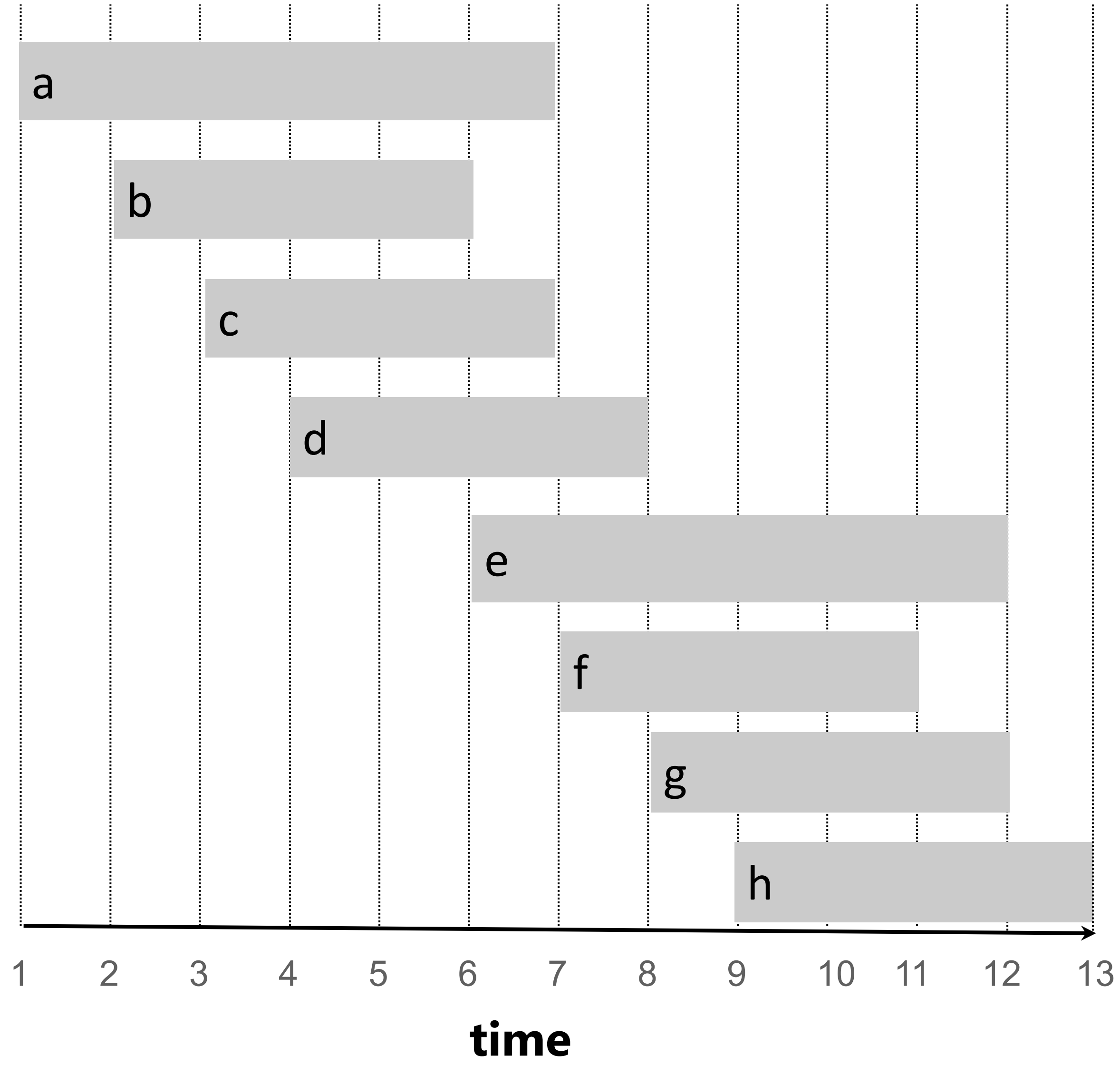


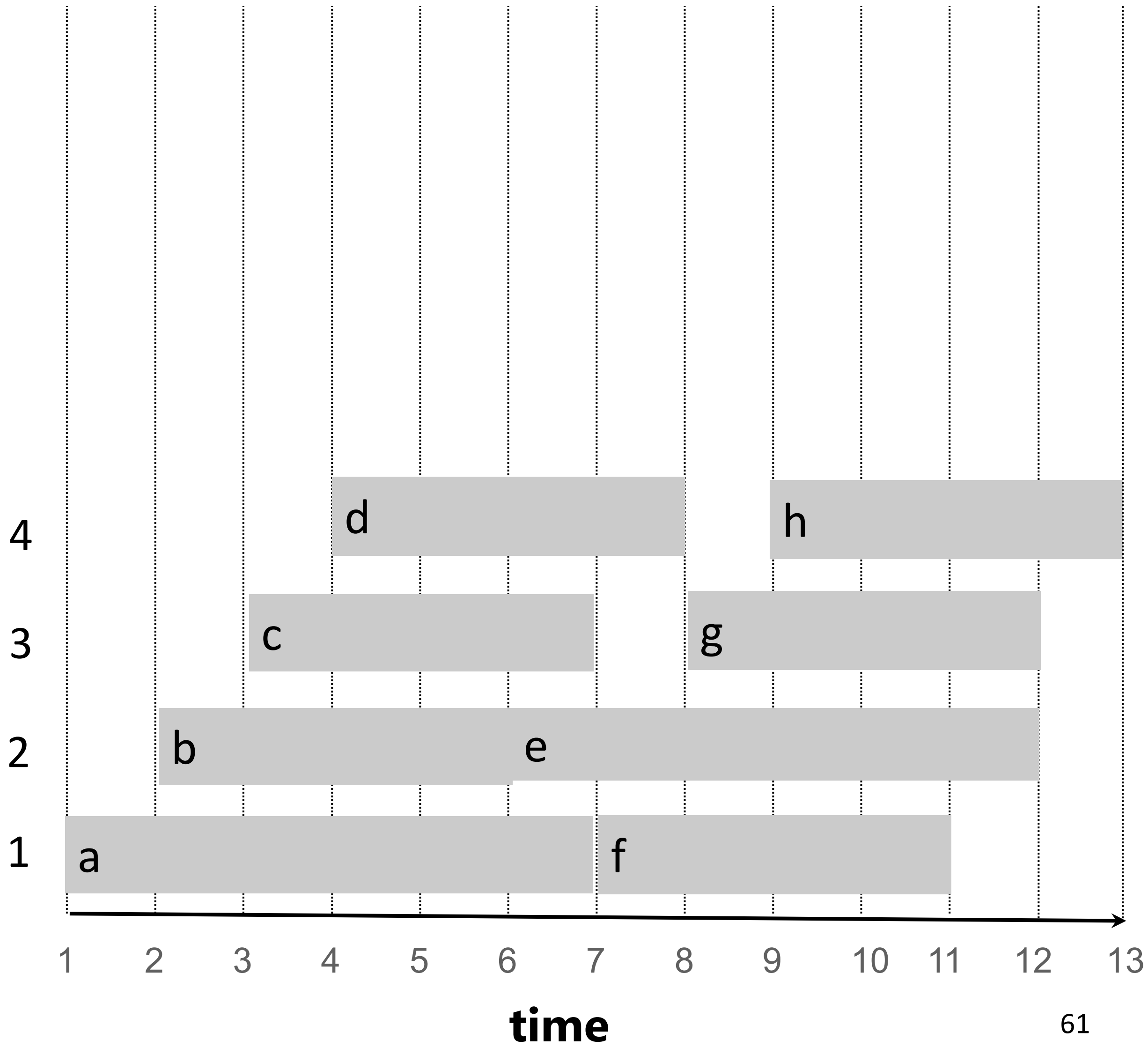
Smallest Interval First





Earliest Start Time First





Interval Partitioning: ESTF Algorithm Analysis

EARLIEST-START-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT lectures by start times and renumber so that $s_1 \leq s_2 \leq \ldots \leq s_n$. $\boldsymbol{\Theta(n \log n)}$

$d \leftarrow 1$ ⟵ number of allocated classrooms

FOR $j = 1$ TO n

  IF (lecture j is compatible with some classroom $k \in \{1, 2, .., d\}$)

    Schedule lecture j in any such classroom k

  ELSE

    Allocate a new classroom $d + 1$.

    Schedule lecture j in classroom $d + 1$.

    $d \leftarrow d + 1$.

RETURN schedule.

$\boldsymbol{\Theta(n)}$

$\boldsymbol{\Theta(\log n)}$

$\boldsymbol{\Theta(n \log n)}$

Interval Partitioning: ESTF Algorithm Analysis

Proposition: The earliest-start-time-first algorithm can be implemented in $O(n \log n)$ time.

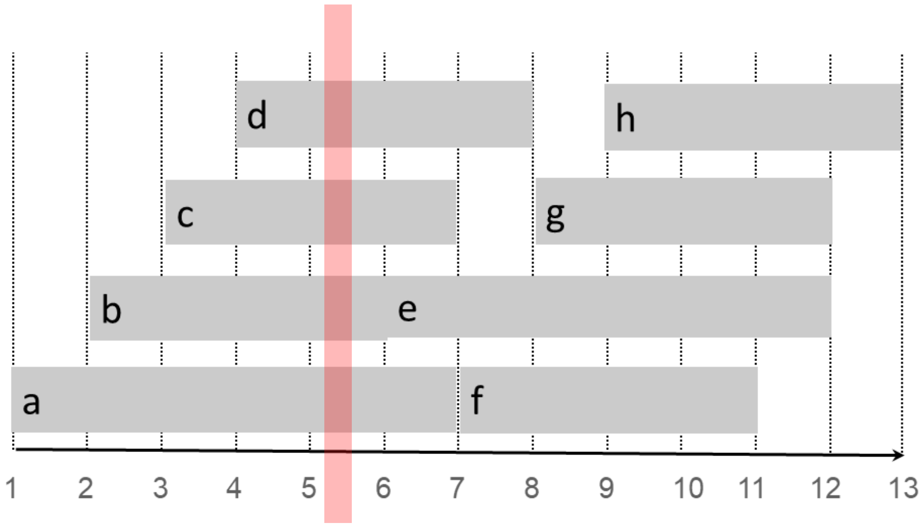
Proof:

- Sorting by start times takes $O(n \log n)$ time.
- Store classrooms in a priority queue (key = finish time of its last lecture).
  - to allocate a new classroom, INSERT classroom onto priority queue.
  - to schedule lecture j in classroom k, INCREASE-KEY of classroom k to f_j.
  - to determine whether lecture j is compatible with any classroom, compare s_j to FIND-MIN
- Total # of priority queue operations is $O(n)$; each takes $O(\log n)$ time. ▪

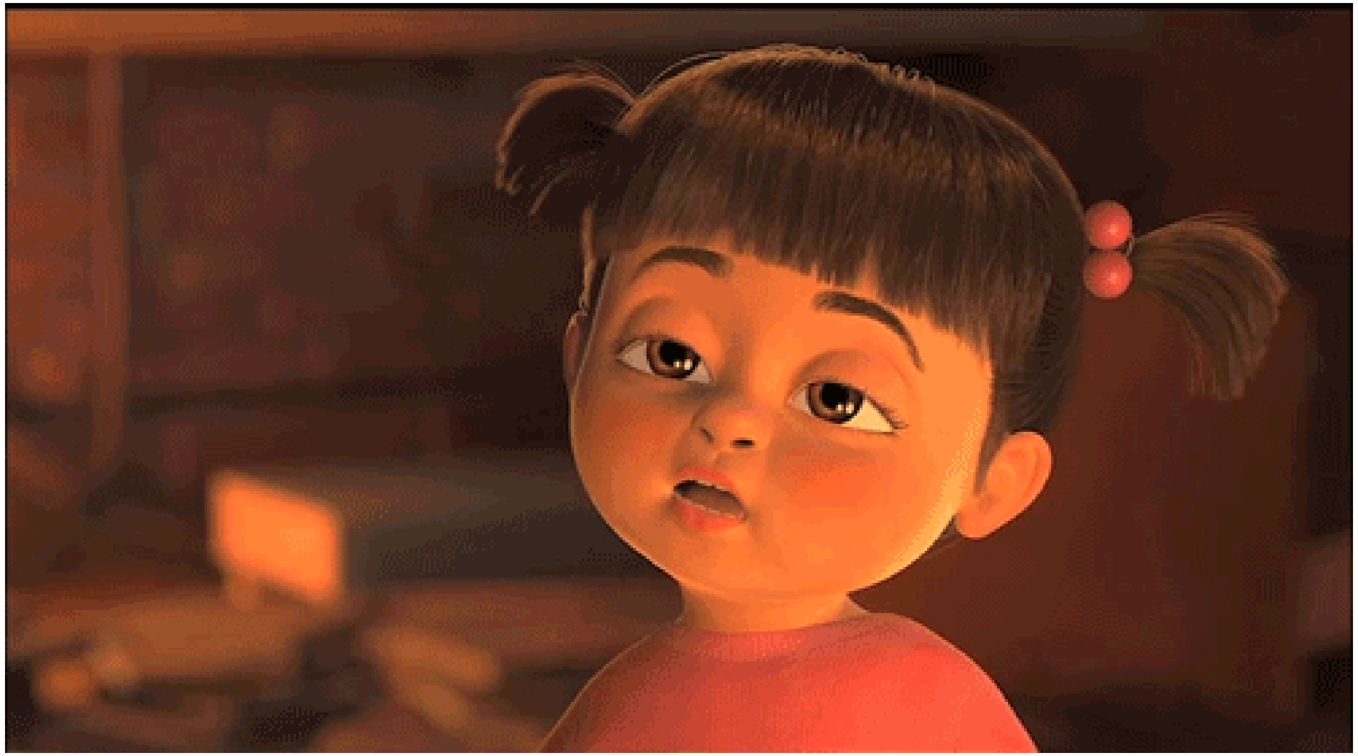
Remark: This implementation chooses a classroom k whose finish time of its last lecture is the earliest.

Interval Partitioning: ESTF Algorithm Analysis

- **Def:** The depth of a set of open intervals is the maximum number of intervals that contain any given point.
- **Key observation:** Number of classrooms needed $\geq$ depth.
- **Q.** Does minimum number of classrooms needed always equal depth?
- **A.** Yes! Moreover, earliest-start-time-first algorithm finds a schedule whose number of classrooms equals the depth.



Thanks a lot



If you are taking a Nap, **wake up**........Lecture Over